Combinatorics of B-orbits and Bruhat–Chevalley order on involutions

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1. Introduction and main results

1.1. Let S_n be the symmetric group on n letters. Bruhat–Chevalley order on S_n is fundamental in a multitude of contexts. For example, it describes the incidences among the closures of double cosets in the Bruhat decomposition of the general linear group $\mathrm{GL}_n(\mathbb{C})$. An interesting subposet of Bruhat–Chevalley order is induced by the involutions, i.e., the elements of order 2 of S_n (we denote this subposet by S_n^2). Activity around S_n^2 was initiated by R. Richardson and T. Springer [RS], who proved that the inverse Bruhat–Chevalley order on S_{2n+1}^2 encodes the incidences among the closed orbits under the action of the Borel subgroup on the symmetric variety $\mathrm{SL}_{2n+1}(\mathbb{C})/\mathrm{SO}_{2n+1}(\mathbb{C})$.

The poset of involutions was also studied by F. Incitti [In1], [In2] from a purely combinatorial point of view. In particular, he proved that this poset is graded, calculated the rank function and described the covering relations. In [BC], E. Bagno and Y. Chernavsky present another geometrical interpretation of the poset S_n^2, considering the action of standard Borel subgroup B (i.e., the group of upper-triangular invertible matrices) of $\mathrm{GL}_n(\mathbb{C})$ on symmetric matrices by congruence. Note that all geometric interpretations deal with the closures of orbits for various actions of the Borel subgroup. The purpose of the paper is to incorporate *coadjoint* orbits into the picture.

Let $\mathfrak{n}$ be the space of strictly upper-triangular matrices and $\mathfrak{n}^*$ its dual space. Since B acts on $\mathfrak{n}$ by conjugation, one can consider the dual action of B on $\mathfrak{n}^*$. To each involution $\sigma \in S_n^2$ one can assign the B-orbit $\Omega_\sigma \subset \mathfrak{n}^*$ (see Subsection 1.2 for precise definitions). Our main result is as follows.

Theorem 1.1. *Let $\sigma, \tau \in S_n^2$. The orbit Ω_τ is contained in the Zariski closure of Ω_σ if and only if $\tau \leq \sigma$ with respect to Bruhat–Chevalley order.*

Note that in [Me1], [Me2], [Me3] A. Melnikov described the incidences among the closures of B-orbits on the variety of upper-triangular 2-nilpotent matrices in combinatorial terms of so-called link patterns and rook placements. (In [BR], M. Boos and M. Reineke generalize the results of Melnikov to all 2-nilpotent matrices; see also B. Rothbach's paper [Ro].) In some sense, our results are "dual" to Melnikov's results.

The paper is organized as follows. In the rest of this Section, we define orbit Ω_σ associated to involution σ from the perspective of representation theory, combinatorics and geometry. Namely, in Subsection 1.2, we give precise definitions and explain the role of orbits Ω_σ in A.A. Kirillov's orbit method in representation theory of the unipotent radical of B. In Subsection 1.3, we briefly recall Melnikov's results and define the partial order $\leq^*$ on S_n^2 in combinatorial terms in the spirit of [Me2]. Then, we formulate Theorem 1.7 claiming that $\leq^*$ encodes the incidences among the closures of Ω_σ, $\sigma \in S_n^2$. In Subsection 1.4, we formulate Theorem 1.10 claiming that the restriction of Bruhat–Chevalley order to S_n^2 coincides with $\leq^*$. Next, in Subsection 1.5, we present a conjectural approach based on the geometry of tangent cones to Schubert varieties.

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In Section 2, we prove Theorem 1.7 (see Propositions 2.3 and 2.10). In Subsection 2.4, using Incitti's results, we prove Theorem 1.10. This concludes the proof of our main result. Section 3 contains the proofs of technical (but important) Lemmas used in the proof of Proposition 2.10. Finally, in Section 4, we present a formula for the dimension of Ω_σ (see Proposition 4.1). We also formulate a conjecture about the closure of Ω_σ and check it in some particular cases (see Subsection 4.2). A short announcement of our results was made in [Ig3].

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1.2. Let $G = \mathrm{GL}_n(\mathbb{C})$ be the general linear group, B its standard Borel subgroup (etc.). Let $U \subset B$ be the *unitriangular* group (i.e., the group of upper-triangular matrices with 1's on the diagonal). Group B acts on $\mathfrak{n}$ by conjugation, so the dual action of B on $\mathfrak{n}^*$ is induced. For $g \in B$ and $\lambda \in \mathfrak{n}^*$, $g.\lambda$ is defined by

$$(g.\lambda)(x) = \lambda(g^{-1}xg), \text{ for } x \in \mathfrak{n}.$$

Let Ω_λ denote the orbit of $\lambda \in \mathfrak{n}^*$ under this action. Let Θ_λ denote the orbit of λ under the action of U. (Clearly, $\Theta_\lambda \subseteq \Omega_\lambda$.)

In 1962, Kirillov showed [Ki1] that there is a bijection between the set $\mathfrak{n}^*/U$ of U-orbits on $\mathfrak{n}^*$ and the set $\widehat{U}$ of equivalence classes of unitary irreducible representations of U in Hilbert spaces. (The proof was adapted for unipotent groups over finite fields by D. Kazhdan in [Ka].) Further, it turned out that all the principle questions about representations can be answered in terms of orbits (see [Ki2] for the details). However, a complete description of $\mathfrak{n}^*/U$ is unknown and seems to be a very difficult problem.

An element $\sigma \in S_n$ satisfying $\sigma^2 = \mathrm{id}$ is called an *involution*. Let S_n^2 be the set of involutions of S_n. To $\sigma \in S_n^2$ one can assign the orbits of the groups B and U by the following rule. Write σ as a product of disjoint cycles: $\sigma = (i_1, j_1) \ldots (i_t, j_t)$, where $i_l > j_l$ for $1 \le l \le t$ and $j_l < j_{l+1}$ for $1 \le l < t$. Denote

$$\Phi = \{(i, j), 1 \le j < i \le n\} \subset \mathbb{Z} \times \mathbb{Z}$$

and put $\mathrm{Supp}(\sigma) = \bigcup_{l=1}^t \{(i_l, j_l)\} \subset \Phi$. Clearly, $\{e_\alpha, \alpha \in \Phi\}$ is a basis of $\mathfrak{n}$. Here $e_\alpha = e_{j,i}$ for $\alpha = (i, j) \in \Phi$, where $e_{j,i}$ is the usual matrix unit. Hence one can consider the dual basis $\{e_\alpha^*, \alpha \in \Phi\}$ of $\mathfrak{n}^*$. Now, to each map $\xi\colon \mathrm{Supp}(\sigma) \to \mathbb{C}^\times \colon \alpha = (i_l, j_l) \mapsto \xi_l$ one can assign the U-orbit $\Theta_{\sigma,\xi}$ by putting $\Theta_{\sigma,\xi} = \Theta_{f_{\sigma,\xi}}$, where

$$f_{\sigma,\xi} = \sum_{\alpha \in \mathrm{Supp}(\sigma)} \xi(\alpha) e_\alpha^* = \sum_{l=1}^t \xi_l e_{j_l, i_l}^*.$$

(If $\sigma = \mathrm{id}$, then $\mathrm{Supp}(\sigma) = \emptyset$ and $f_{\sigma,\xi} = 0$.) We say that $\Theta_{\sigma,\xi}$ is *associated* with σ and ξ. Set $\xi_0(\alpha) = 1$ for all $\alpha \in \mathrm{Supp}(\sigma)$, and $\Omega_\sigma = \Omega_{f_{\sigma,\xi_0}}$. (In other words, $f_{\sigma,\xi_0} = \sum_{\alpha \in \mathrm{Supp}(\sigma)} e_\alpha^*$.) Lemma 2.1 shows that $\Omega_\sigma = \bigcup \Theta_{\sigma,\xi}$, where the union is taken over all maps $\xi\colon \mathrm{Supp}(\sigma) \to \mathbb{C}^\times$.

It turned out that almost all U-orbits on $\mathfrak{n}^*$ studied so far are associated with involutions.

Example 1.2. i) Being an orbit of a connected unipotent group on an affine variety, any U-orbit is a Zariski-closed irreducible subvariety of $\mathfrak{n}^*$. Let Θ be an orbit of maximal dimension (such an orbit is called *regular*). Then either $\Theta = \Theta_{w_0,\xi}$ or $\Theta = \Theta_{w_1,\xi}$ for some ξ (in the last case n must be even). Here $w_0 = (n, 1)(n-1, 2) \ldots (n-n_0+1, n_0)$, $n_0 = [n/2]$, and $\mathrm{Supp}(w_1) = \mathrm{Supp}(w_0) \setminus \{(n-n_0+1, n_0)\}$. Conversely, all $\Theta_{w_1,\xi}$'s are regular [Ki1, §9, Example 2].

ii) An orbit $\Theta \subset \mathfrak{n}^*$ is called *subregular* if it has the second maximal dimension. Pick $1 \leq j < n_0$ and put σ to be the involution such that

$$\mathrm{Supp}(\sigma) = (\mathrm{Supp}(w_0) \setminus \{(n-j+1, j), (n-j, j+1)\}) \cup \{(n-j+1, j+1), (n-j, j)\}.$$

Then $\Theta_{\sigma,\xi}$ is subregular for all ξ. Subregular orbits were described by Panov in [Pa].

iii) Let $\alpha = (i, j) \in \Phi$. The orbit of e_α^* is called *elementary*. Evidently, it is associated with the involution $\sigma = (i, j) \in S_n^2$. Elementary orbits are described in [Mu].

Thus, orbits associated with involutions play an important role in representation theory. (See [An1], [An2], [AN], [Ig1] and [Ig2] for further examples and generalizations to other unipotent algebraic groups.) They were completely described by Panov in [Pa]. In particular, for a given orbit $\Theta_{\sigma,\xi}$, he presented the set of equations defining this orbit as a closed subvariety of $\mathfrak{n}^*$. On the contrary, B-orbits Ω_σ are *not* closed, so the natural question arises: given two orbits Ω_τ and Ω_σ, $\sigma, \tau \in S_n^2$, when $\Omega_\tau \subseteq \overline{\Omega}_\sigma$? (Here $\overline{Z}$ denotes the Zariski closure of a subset $Z \subseteq \mathfrak{n}^*$.) By Theorem 1.1, this occurs if and only if $\tau \leq_B \sigma$, where $\leq_B$ denotes Bruhat–Chevalley order.

1.3. Let $\mathcal{N} \subset \mathfrak{n}$ be the variety of upper-triangular matrices of square zero:

$$\mathcal{N} = \{X \in \mathfrak{n} \mid X^2 = 0\}.$$

Group B acts on $\mathcal{N}$ by conjugation. For a given $X \in \mathcal{N}$, let $\mathcal{O}_X$ denote the orbit of X under this action. To $\sigma \in S_n^2$ one can assign the orbit $\mathcal{O}_\sigma$ by the following rule. Write σ as a product of disjoint cycles: $\sigma = (i_1, j_1) \ldots (i_t, j_t)$, where $i_l > j_l$ for $1 \leq l \leq t$ and $j_l < j_{l+1}$ for $1 \leq l < t$. Denote by $X_\sigma \in \mathcal{N}$ the matrix of the form $X_\sigma = \sum_{\alpha \in \mathrm{Supp}(\sigma)} e_\alpha = \sum_{l=1}^{t} e_{j_l, i_l}$, and put $\mathcal{O}_\sigma = \mathcal{O}_{X_\sigma}$. By [Me1, Theorem 2.2], one has

$$\mathcal{N} = \bigsqcup_{\sigma \in S_n^2} \mathcal{O}_\sigma.$$

To each $\sigma \in S_n^2$ one can also assign the matrix R_σ by putting

$$(R_\sigma)_{i,j} = \mathrm{rk}\, \pi_{i,j}(X_\sigma),$$

where $\pi_{i,j} \colon \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$ acts on a matrix by replacing all entries of the first $(i-1)$ rows and the last $(n-j)$ columns by zeroes. Let us define a partial order on S_n^2. Given $\sigma, \tau \in S_n^2$, we put $\sigma \leq \tau$ if $R_\sigma \leq R_\tau$, i.e., $(R_\sigma)_{i,j} \leq (R_\tau)_{i,j}$ for all $1 \leq i < j \leq n$.

Example 1.3. Let $n = 5$, $\sigma = (3,1)(5,2)$, $\tau = (2,1)(4,3) \in S_5^2$. Then

$$X_\sigma = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_\tau = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_\sigma = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_\tau = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so $R_\sigma \leq R_\tau$ and $\sigma \leq \tau$.

Remark 1.4. Note that this partial order has an interpretation in terms of so-called *rook placements*. Namely, X_σ can be treated as a rook placement on the triangle board with boxes labeled by pairs (i, j), $1 \leq i < j \leq n$: by definition, there is a rook in the (i, j)th box if and only if $(X_\sigma)_{i,j} = 1$. Then $(R_\sigma)_{i,j}$ is just the number of rooks located non-strictly to the South-West of the (i, j)th box.

As above, let $\overline{Z}$ be the closure of a subset $Z \subseteq \mathrm{Mat}_n(\mathbb{C})$ with respect to Zariski topology. By [Me2, Theorem 3.5], one has the following nice combinatorial description of the orbit closures in $\mathcal{N}$:

$$\mathcal{O}_\tau \subseteq \overline{\mathcal{O}}_\sigma \text{ if and only if } \tau \leq \sigma. \tag{1}$$

In [Me3], an interpretation of this result in terms of link patterns is given.

Now, let $\mathfrak{n}_-$ be the space of strictly lower-triangular matrices (with zeroes on the diagonal). We can identify it with $\mathfrak{n}^*$ by putting

$$\lambda(x) = \langle \lambda, x \rangle = \operatorname{tr} \lambda x,\ \ \lambda \in \mathfrak{n}_-,\ \ x \in \mathfrak{n}.$$

Thus, in the sequel we identify $\mathfrak{n}^*$ with $\mathfrak{n}_-$. Note that under this identification, $e_\alpha^* = e_{i,j}$ for all $\alpha = (i, j) \in \Phi$, and $\Omega_\lambda = \{(g\lambda g^{-1})_{\mathrm{low}}, g \in B\}$, where A_{low} denotes the strictly lower-triangular part of A, that is

$$(A_{\mathrm{low}})_{i,j} = \begin{cases} A_{i,j}, & \text{if } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\sigma \in S_n^2$. Then f_{σ,ξ_0} is identified with X_σ^t, so Ω_σ is identified with $\Omega_{X_\sigma^t}$, where $X_\sigma^t \in \mathfrak{n}^*$ denotes the transposed matrix to X_σ. In fact, our goal is to describe $\overline{\Omega}_\sigma$ in combinatorial terms. To do this, let us define another partial order on S_n^2. Given σ, $\tau \in S_n^2$, put $\sigma \leq^* \tau$ if $R_\sigma^* \leq R_\tau^*$, i.e., $(R_\sigma^*)_{i,j} \leq (R_\tau^*)_{i,j}$ for all $1 \leq j < i \leq n$. Here $R_\sigma^* \in \mathfrak{n}^*$ is the matrix defined by the rule

$$(R_\sigma^*)_{i,j} = \begin{cases} \operatorname{rk} \pi_{i,j}(X_\sigma^t), & \text{if } 1 \leq j < i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

As above, $\pi_{i,j} \colon \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$ acts on a matrix by replacing all entries of the first $(i-1)$ rows and the last $(n-j)$ columns by zeroes.

Example 1.5. Let $n = 5$, $\sigma = (4, 1)(5, 2)$, $\tau = (5, 1)(4, 2) \in S_5^2$. Then

$$X_\sigma^t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},\ X_\tau^t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_\sigma^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix},\ R_\tau^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

so $R_\sigma^* \leq R_\tau^*$ and $\sigma \leq^* \tau$.

Remark 1.6. Of course, this partial order has an interpretation in terms of rook placements. Namely, X_σ^t can be treated as a rook placement on the triangle board with boxes labeled by pairs (i, j), $1 \leq j < i \leq n$: by definition, there is a rook in the (i, j)th box if and only if $(X_\sigma^t)_{i,j} = 1$. Then $(R_\sigma^*)_{i,j}$, $i > j$, is just the number of rooks located non-strictly to the South-West of the (i, j)th box.

The following theorem plays a key role in the proof of the main result of the paper (cf. (1)).

Theorem 1.7. *Let σ, τ be involutions in S_n and $\Omega_\sigma, \Omega_\tau$ the corresponding B-orbits in $\mathfrak{n}^*$. Then*

$$\Omega_\tau \subseteq \overline{\Omega}_\sigma \textit{ if and only if } \tau \leq^* \sigma.$$

The proof will be presented in the next Section (see Proposition 2.3 for the proof of "only if" direction and Proposition 2.10 for the proof of "if" direction).

Remark 1.8. Note that there is *no* natural analogue of the variety $\mathcal{N}$ in the space $\mathfrak{n}^*$. Actually, one can put

$$\mathcal{N}^* = \bigsqcup_{\sigma\in S_n^2} \Omega_\sigma.$$

Clearly, this subset of $\mathfrak{n}^*$ is stable under the action of B, but it is neither open nor closed, if $n > 2$. (For $n = 2$, $\mathcal{N} = \mathfrak{n}^*$.) Indeed, it contains the orbit Ω_{w_0}, where $w_0 = (n, 1)(n-1, 2)\dots(n-n_0+1, n_0)$, $n_0 = [n/2]$ (as in Example 1.2i)). It follows from [Ki1, §9, Example 2] and Lemma 2.1 that $y \in \mathfrak{n}^*$ belongs to Ω_{w_0} if and only if $\Delta_i(y) \neq 0$ for all $1 \leq i \leq n_0$. Here

$$\Delta_i(y) = \begin{vmatrix} y_{n-i+1,1} & y_{n-i+1,2} & \cdots & y_{n-i+1,i} \\ y_{n-i+2,1} & y_{n-i+2} & \cdots & y_{n-i+2,i} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1} & y_{n,2} & \cdots & y_{n,i} \end{vmatrix}.$$

Hence Ω_{w_0} is an open subset of $\mathfrak{n}^*$, so $\overline{\mathcal{N}^*} = \overline{\Omega}_{w_0} = \mathfrak{n}^*$ and $\mathcal{N}^*$ is not closed.

On the other hand, suppose $\mathcal{N}^*$ is open. Consider $V = \{y \in \mathfrak{n}^* \mid y_{i,j} = 0, \text{ if } i > 3\}$. Then $\mathcal{N}^* \cap V$ must be an open subset of V. However, Lemma 2.1 together with [Pa, Theorem 1.4] imply that

$$\mathcal{N}^* \cap V = \{y \in V \mid y_{3,1} \neq 0\} \cup \{y \in V \mid y_{2,1} = y_{3,1} = 0\} \cup \{y \in V \mid y_{3,1} = y_{3,2} = 0\},$$

which is obviously not an open subset of V, a contradiction. Note, however, that $\mathcal{N}^*$ is an irreducible constructive subset of $\mathfrak{n}^*$ (as a union of orbits containing a dense subset of $\mathfrak{n}^*$). Note also that, unlike of the adjoint case considered by Melnikov, the closure of a given Ω_σ, $\sigma \in S_n^2$, is *not* a subset of $\mathcal{N}^*$ (see Subsection 4.2 for a conjectural description of $\overline{\Omega}_\sigma$).

1.4. Recall that the *Bruhat–Chevalley order* $\leq_B$ on S_n is defined in terms of the inclusion relationships of double cosets in $\mathrm{GL}_n(\mathbb{C})$. Namely, $G = \mathrm{GL}_n(\mathbb{C}) = \bigcup_{w\in S_n} B\dot{w}B$, where $\dot{w}$ denotes the permutation matrix corresponding to w. Let $v, w \in S_n$. By definition, $v \leq_B w$ if $B\dot{v}B \subseteq \overline{B\dot{w}B}$. Let $w = s_1 \dots s_l$ be a *reduced* expression of w as a product of *simple reflections* $s_i = (i, i+1) \in S_n$, $1 \leq i \leq n-1$, and $l(w) = l$. It's well-known that

$$\{v \in S_n \mid v \leq_B w\} = \{s_{i_1} \dots s_{i_k},\ 1 \leq i_1 < \dots < i_k \leq l\}.$$

Further, let $X \in \mathrm{Mat}_n(\mathbb{C})$ be an arbitrary 0–1 matrix with at most one 1 in every row and every column. Denote by $R(X)$ the matrix such that

$$R(X)_{i,j} = \mathrm{rk}\,\pi_{i,j}(X),\ 1 \leq i, j \leq n$$

(see the previous Subsection for the definition of $\pi_{i,j}$).

Remark 1.9. Notice that $R(X)_{i,j}$ is just the number of rooks located non-strictly to the South-West of the (i, j)th box. As above, for a given matrix $A \in \mathrm{Mat}_n(\mathbb{C})$, let A_{low} denote the strictly lower-triangular part of A. Then $R^*_\sigma = R(\dot{\sigma})_{\mathrm{low}}$.

Let v, $w \in S_n$. Then (see, e.g., [Pr])

$$v \leq_B w \text{ if and only if } R(\dot{v}) \leq R(\dot{w}), \text{i.e., } R(\dot{v})_{i,j} \leq R(\dot{w})_{i,j} \text{ for all } 1 \leq i, j \leq n.$$

Suppose σ, $\tau \in S_n^2$. It follows immediately from Remark 1.9 that $\tau \leq_B \sigma$ implies $\tau \leq^* \sigma$. In fact, the second ingredient of the proof of Theorem 1.1 is the fact that these conditions are equivalent, i.e., the order on S_n^2 induced by Bruhat–Chevalley order coincides with $\leq^*$.

Theorem 1.10. *Let σ, τ be involutions in S_n. Then*

$$\tau \leq^* \sigma \text{ if and only if } \tau \leq_B \sigma.$$

The proof based on the computing the covering relations for $\leq^*$ and on Incitti's results is presented in Subsection 2.4. Note that this Theorem together with Theorem 1.7 imply our main result.

1.5. Before starting with the proof of Theorem 1.1, we will briefly describe another (conjectural) approach to orbits associated with involutions in terms of tangent cones to Schubert varieties. Since $G = \bigcup_{w \in S_n} B\dot{w}B$, the *flag variety* $\mathcal{F} = G/B$ can be decomposed into the union $\mathcal{F} = \bigcup_{w \in S_n} X_w^\circ$, where $X_w^\circ = B\dot{w}B/B$ is called the *Schubert cell.* By definition, the *Schubert variety* X_w is the closure of X_w° in $\mathcal{F}$ with respect to Zariski topology. Note that $p = X_{\mathrm{id}} = B/B$ is contained in X_w for all $w \in S_n$. One has $X_w \subseteq X_{w'}$ if and only if $w \leq_B w'$. Let T_w be the tangent space and C_w the tangent cone to X_w at the point p (see [BL] for detailed constructions); by definition, $C_w \subseteq T_w$ and if p is a regular point of X_w, then $C_w = T_w$. Of course, if $w \leq_B w'$, then $C_w \subseteq C_{w'}$.

Let $T = T_p\mathcal{F}$ be the tangent space to $\mathcal{F}$ at p. It can be naturally identified with $\mathfrak{n}^*$ in the following way. Since $\mathcal{F} = G/B$, T is isomorphic to the factor $\mathfrak{g}/\mathfrak{b}$, where $\mathfrak{g} = \mathrm{Mat}_n(\mathbb{C})$ is the Lie algebra of G and $\mathfrak{b} = \langle e_{i,j}, 1 \leq i \leq j \leq n\rangle_{\mathbb{C}}$ is the Lie algebra of B. In turn, $\mathfrak{g}/\mathfrak{b}$ is naturally isomorphic to $\mathfrak{n}_- = \mathfrak{n}^*$. Next, B acts on $\mathcal{F}$ by left multiplications. Since p is invariant under this action, the action on $T = \mathfrak{n}^*$ is induced. One can easily check that this action coincides with the action of B on $\mathfrak{n}^*$ defined above [Ki3, Section 3, Theorem 1]. Further, the tangent cone $C_w \subseteq T_w \subseteq T = \mathfrak{n}^*$ is B-invariant, so it splits into a union of B-orbits.

It is well-known that C_w is a subvariety of T_w of dimension $\dim C_w = l(w)$ [BL, Chapter 2, Section 2.6]. Let $\sigma \in S_n^2$. $\overline{\Omega}_\sigma$ is irreducible as the closure of an orbit. By Proposition 4.1, $\dim \overline{\Omega}_\sigma = \dim \Omega_\sigma = l(\sigma)$, so $\overline{\Omega}_\sigma$ is an irreducible component of C_σ of maximal dimension. For $n \leq 5$, $C_\sigma = \overline{\Omega}_\sigma$ for all $\sigma \in S_n^2$. (See [EP] for an explicit description of tangent cones.) Unfortunately, we can not prove the irreducibility of C_σ for all $\sigma \in S_n^2$ for an arbitrary n. On the other hand, we do not know counterexamples to the equality $C_\sigma = \overline{\Omega}_\sigma$. This allows us to formulate

Conjecture 1.11. *Let $\sigma \in S_n$ be an involution. Then the closure of the B-orbit $\Omega_\sigma \subset \mathfrak{n}^*$ coincides with the tangent cone C_w to the Schubert variety X_w at the point $p = B/B$.*

Note that this conjecture implies that if $\tau \leq_B \sigma$, then $\Omega_\tau \subseteq \overline{\Omega}_\sigma$.

2. Proof of the Main Theorem

2.1. The goal of this Subsection is to prove the "only if" direction of Theorem 1.7. Fix an involution $\sigma \in S_n^2$. Recall notation from Subsection 1.2. Let $D \subset B$ be the subgroup of diagonal matrices. Clearly, $B = U \rtimes D$. (In other words, for a given $g \in B$, there exist unique $u \in U$, $d \in D$ such that $g = ud$.)

Lemma 2.1. *One has*[1] $\Omega_\sigma = \bigcup_{\xi\colon \mathrm{Supp}(\sigma) \to \mathbb{C}^\times} \Theta_{\sigma,\xi}$.

Proof. Let $\xi\colon \mathrm{Supp}(\sigma) \to \mathbb{C}^\times$ be a map. If $d = 1_n + \sum_{l=1}^t (\xi_l - 1)e_{i_l,i_l} \in D$, then $d.X_\sigma^t = f_{\sigma,\xi}$, so $\Theta_{\sigma,\xi} \subset \Omega_\sigma$. On the other hand, let g be an element of B. Then there exist $u \in U$, $d \in D$ such that $g = ud$, so $g.X_\sigma^t = u.f_{\sigma,\xi}$, where $\xi(i_l, j_l) = g_{i_l,i_l}/g_{j_l,j_l}$. Thus, $g.X_\sigma^t \in \Theta_{\sigma,\xi}$. □

Lemma 2.2. *Let $\lambda \in \Omega_\sigma$. Then $\mathrm{rk}\,\pi_{i,j}(\lambda) = (R_\sigma^*)_{i,j}$ for all $1 \leq j < i \leq n$.*

Proof. Fix a map $\xi\colon \mathrm{Supp}(\sigma) \to \mathbb{C}^\times$. Lemma 2.1 shows that it's enough to check that if $u \in U$, $f \in \mathfrak{n}^*$, then $\mathrm{rk}\,\pi_{i,j}(u.f) = \mathrm{rk}\,\pi_{i,j}(f)$ for all $1 \leq j < i \leq n$, because $\mathrm{rk}\,\pi_{i,j}(f_{\sigma,\xi}) = \mathrm{rk}\,\pi_{i,j}(X_\sigma^t) = (R_\sigma^*)_{i,j}$. Pick an element $u \in U$. It's well-known that there exist $\alpha_{j,i} \in \mathbb{C}$ such that

$$u = \prod_{(i,j) \in \Phi} x_{j,i}(\alpha_{j,i}),$$

[1] Cf. [Me1, Subsection 3.3].

where $x_{j,i}(\alpha_{j,i}) = 1_n + \alpha_{j,i}e_{j,i}$ (the product is taken in any fixed order). Hence we can assume $u = x_{j,i}(\alpha)$ for some $(i,j) \in \Phi$, $\alpha \in \mathbb{C}$. Then

$$(u.f)_{r,s} = \begin{cases} f_{j,s} + \alpha f_{i,s}, & \text{if } r = j \text{ and } 1 \le s < j,\\ f_{r,i} - \alpha f_{r,j}, & \text{if } s = i \text{ and } i < r \le n,\\ f_{r,s} & \text{otherwise.} \end{cases}$$

Hence if $r > j$ and $s < i$, then $\pi_{r,s}(u.f) = \pi_{r,s}(f)$. If $r \le j$ (and so $s < r \le j < i$), then the jth row of $\pi_{r,s}(u.f)$ is obtained from the jth row of $\pi_{r,s}(f)$ by adding the ith row of $\pi_{r,s}(f)$ multiplied by α. Similarly, if $s \ge i$ (and so $r > s \ge i > j$), then the ith column of $\pi_{r,s}(u.f)$ is obtained from the ith column of $\pi_{r,s}(f)$ by subtracting the jth column of $\pi_{r,s}(f)$ multiplied by α. In both cases, $\operatorname{rk}\pi_{r,s}(u.f) = \operatorname{rk}\pi_{r,s}(f)$, as required. $\square$

Proposition 2.3. *Let σ, τ be involutions in S_n. If $\Omega_\tau \subseteq \overline{\Omega}_\sigma$, then[2] $\tau \le^* \sigma$.*

PROOF. Suppose $\sigma \not\ge^* \tau$, This means that there exists $(i,j) \in \Phi$ such that $(R^*_\sigma)_{i,j} < (R^*_\tau)_{i,j}$. Denote

$$Z = \{f \in \mathfrak{n}^* \mid \operatorname{rk}\pi_{r,s}(f) \le (R^*_\sigma)_{r,s} \text{ for all } (r,s) \in \Phi\}.$$

Clearly, Z is closed with respect to Zariski topology. Lemma 2.2 shows that $\Omega_\sigma \subseteq Z$, so $\overline{\Omega}_\sigma \subseteq Z$. But $X^t_\tau \notin Z$, hence $\Omega_\tau \not\subseteq Z$, a contradiction. $\square$

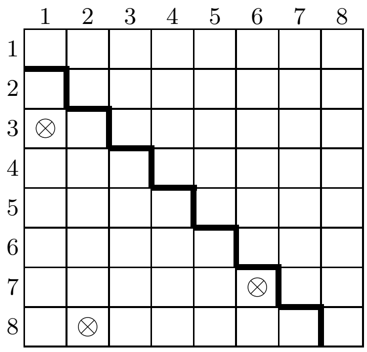
2.2. Now, let us start with the proof of much more difficult "if" direction of Theorem 1.7. First, we need some more notation (cf. [Me2, Subsections 3.7–3.14]). There exists a natural partial order on Φ. Namely, given (a,b), $(c,d) \in \Phi$, we set $(a,b) \le (c,d)$ if $a \le c$ and $b \ge d$; we also set $(a,b) > (c,d)$, if $(a,b) \ge (c,d)$ and $(a,b) \ne (c,d)$. Let $\sigma \in S^2_n$ and $(i,j) \in \operatorname{Supp}(\sigma)$, i.e., $i > j$ and $\sigma(i) = j$. Denote

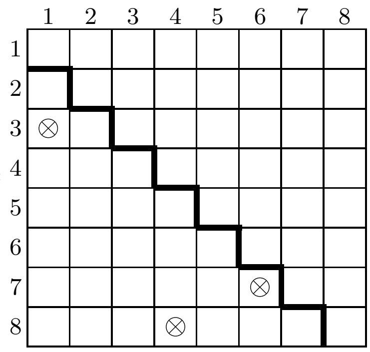
$$m = \min\{s \mid j < s < i \text{ and } \sigma(s) = s\}.$$

Suppose m exists. Further, suppose that there are *no* $(p,q) \in \operatorname{Supp}(\sigma)$ such that $(i,j) > (p,q)$, $(i,m) \not> (p,q)$. Then denote by $\sigma_{(i,j)}^{\to} \in S^2_n$ the involution such that

$$\operatorname{Supp}(\sigma_{(i,j)}^{\to}) = (\operatorname{Supp}(\sigma) \setminus \{(i,j)\}) \cup \{(i,m)\}.$$

Example 2.4. It's very convenient to draw the corresponding X's as rook placements. For example, if $n = 8$, $\sigma = (3,1)(8,2)(7,6)$, then $\sigma_{(8,2)}^{\to} = (3,1)(8,4)(7,6)$, so

$X^t_\sigma =$ 

and $X^t_{\sigma_{(8,2)}^{\to}} =$ 

Here we denote rooks by $\otimes$'s.

Similarly, suppose

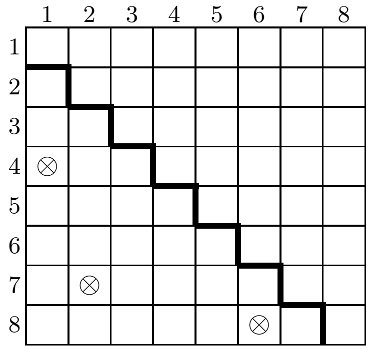
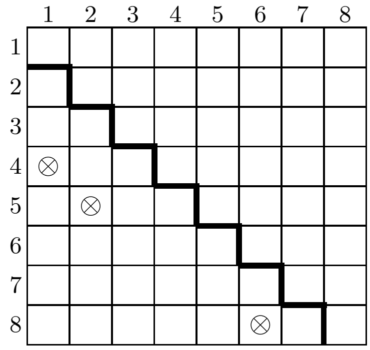
$$m = \max\{r \mid j < r < i \text{ and } \sigma(r) = r\}$$

[2] Cf. [Me2, Lemma 3.6].

exists. Further, suppose that there are no $(p,q) \in \mathrm{Supp}(\sigma)$ such $(i,j) > (p,q)$, $(m,j) \not> (p,q)$. Then denote by $\sigma^{\uparrow}_{(i,j)} \in S_n^2$ the involution such that

$$\mathrm{Supp}(\sigma^{\uparrow}_{(i,j)}) = (\mathrm{Supp}(\sigma) \setminus \{(i,j)\}) \cup \{(m,j)\}.$$

Example 2.5. Let $n = 8$, $\sigma = (4,1)(7,2)(8,6)$, then $\sigma^{\uparrow}_{(7,2)} = (4,1)(5,2)(8,6)$, so

$X^t_{\sigma} =$  and $X^t_{\sigma^{\uparrow}_{(7,2)}} =$ 

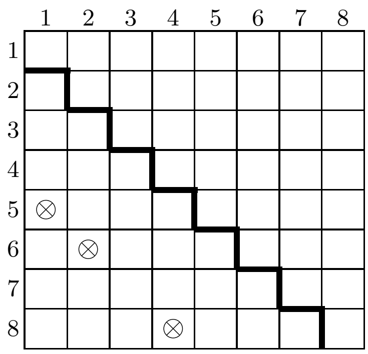
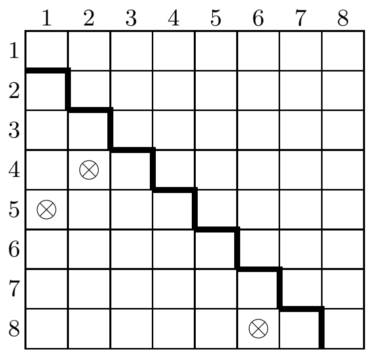
Now, denote by $M(\sigma)$ the set of minimal elements of $\mathrm{Supp}(\sigma)$ with respect to the partial order defined above. If $(i,j) \in M(\sigma)$, then we denote also by $\sigma^{-}_{(i,j)} \in S_n^2$ the involution such that

$$\mathrm{Supp}(\sigma^{-}_{(i,j)}) = \mathrm{Supp}(\sigma) \setminus \{(i,j)\}.$$

Then, denote by $A_{i,j}(\sigma)$, $(i,j) \in \mathrm{Supp}(\sigma)$, the set of $(\alpha,\beta) \in \mathrm{Supp}(\sigma)$ such that $j < \beta < i < \alpha$, $\sigma(r) \neq r$ for all $\beta < r < i$, and there are no $(p,q) \in \mathrm{Supp}(\sigma)$ such that either $j < q < \beta < p < i$ or $\beta < q < i < p < \alpha$ (i.e., either $(i,j) > (p,q)$ and $(\beta,j) \not> (p,q)$, or $(\alpha,\beta) > (p,q)$ and $(\alpha,i) \not> (p,q)$). If $(\alpha,\beta) \in A_{i,j}(\sigma)$, then denote by $a\sigma^{(\alpha,\beta)}_{(i,j)} \in S_n^2$ the involution such that

$$\mathrm{Supp}(a\sigma^{(\alpha,\beta)}_{(i,j)}) = (\mathrm{Supp}(\sigma) \setminus \{(i,j),(\alpha,\beta)\}) \cup \{(\beta,j),(\alpha,i)\}.$$

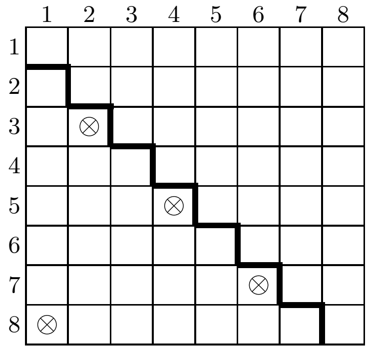
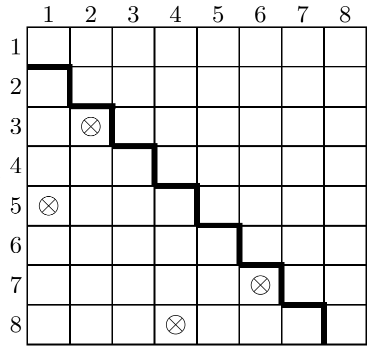
Example 2.6. If $n = 8$, $\sigma = (5,1)(6,2)(8,4)$, then $(8,4) \in A_{6,2}(\sigma)$, $a\sigma^{(8,4)}_{(6,2)} = (5,1)(4,2)(8,6)$, so

$X^t_{\sigma} =$  and $X^t_{a\sigma^{(8,4)}_{(6,2)}} =$ 

Now, denote by $B_{i,j}(\sigma)$, $(i,j) \in \mathrm{Supp}(\sigma)$, the set of $(\alpha,\beta) \in \mathrm{Supp}(\sigma)$ such that $(\alpha,\beta) > (i,j)$ and there are no $(p,q) \in \mathrm{Supp}(\sigma)$ such that $(i,j) < (p,q) < (\alpha,\beta)$. If $(\alpha,\beta) \in B_{i,j}(\sigma)$, then denote by $b\sigma^{(\alpha,\beta)}_{(i,j)} \in S_n^2$ the involution such that

$$\mathrm{Supp}(b\sigma^{(\alpha,\beta)}_{(i,j)}) = (\mathrm{Supp}(\sigma) \setminus \{(i,j),(\alpha,\beta)\}) \cup \{(i,\beta),(\alpha,j)\}.$$

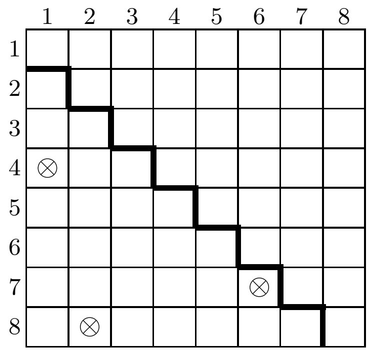
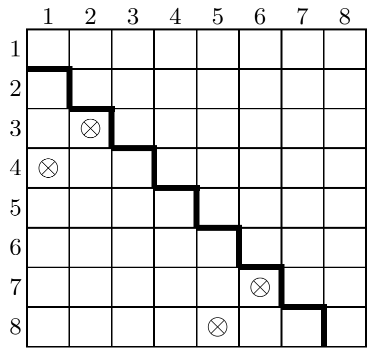
Example 2.7. Let $n = 8$ and $\sigma = (8,1)(3,2)(5,4)(7,6)$. In this case, $(8,1) \in B_{5,4}(\sigma)$ and $b\sigma^{(8,1)}_{(5,4)} = (5,1)(3,2)(8,4)(7,6)$, so

$X^t_\sigma =$  and $X^t_{b\sigma^{(8,1)}_{(5,4)}} =$ 

Finally, denote by $C_{i,j}(\sigma)$, $(i,j) \in \mathrm{Supp}(\sigma)$, the set of $(\alpha,\beta) \in \mathbb{Z}\times\mathbb{Z}$ such that $i > \beta > \alpha > j$, $\sigma(s) \neq s$ for all $\beta > s > \alpha$, and if $(p,q) \in \mathrm{Supp}(\sigma)$, $(i,j) > (p,q)$, $(\alpha,j) \not> (p,q)$, then $(i,\beta) > (p,q)$. If $(\alpha,\beta) \in C_{i,j}(\sigma)$, then denote by $c\sigma^{\alpha,\beta}_{(i,j)} \in S^2_n$ the involution such that

$$\mathrm{Supp}(c\sigma^{\alpha,\beta}_{(i,j)}) = (\mathrm{Supp}(\sigma) \setminus \{(i,j)\}) \cup \{(i,\beta),(\alpha,j)\}.$$

Example 2.8. If $n = 8$, $\sigma = (4,1)(8,2)(7,6)$, then $(3,5) \in C_{8,2}(\sigma)$, $c\sigma^{3,5}_{(8,2)} = (4,1)(3,2)(8,5)(7,6)$, so one has

$X^t_\sigma =$  and $X^t_{c\sigma^{3,5}_{(8,2)}} =$ 

2.3. Let $\sigma \in S^2_n$. Put $\mathrm{Near}(\sigma) = N^+(\sigma) \cup N^-(\sigma) \cup N^0(\sigma)$, where

$$\begin{aligned}
N^-(\sigma) &= \{\sigma^-_{(i,j)}, (i,j) \in M(\sigma)\},\\
N^+(\sigma) &= \{c\sigma^{\alpha,\beta}_{(i,j)}, (i,j) \in \mathrm{Supp}(\sigma), (\alpha,\beta) \in C_{i,j}(\sigma)\},\\
N^0(\sigma) &= \{a\sigma^{(\alpha,\beta)}_{(i,j)}, (i,j) \in \mathrm{Supp}(\sigma), (\alpha,\beta) \in A_{i,j}(\sigma)\}\\
&\quad \cup \{b\sigma^{(\alpha,\beta)}_{(i,j)}, (i,j) \in \mathrm{Supp}(\sigma), (\alpha,\beta) \in B_{i,j}(\sigma)\}\\
&\quad \cup \{\sigma^{\rightarrow}_{(i,j)}, (i,j) \in \mathrm{Supp}(\sigma)\} \cup \{\sigma^{\uparrow}_{(i,j)}, (i,j) \in \mathrm{Supp}(\sigma)\}.
\end{aligned}$$

Proposition 2.9. *Let $\sigma \in S^2_n$. If $\tau \in \mathrm{Near}(\sigma)$, then*[3] $\Omega_\tau \subset \overline{\Omega}_\sigma$.

Proof. Denote by $\overline{Z}^{\mathbb{C}} \subset \overline{Z}$ the closure of a subset $Z \subseteq \mathfrak{n}^*$ in the complex topology. It's well-known that $\overline{\Omega}^{\mathbb{C}}_\sigma = \overline{\Omega}_\sigma$, so it's enough to prove that $X^t_\tau \in \overline{\Omega}^{\mathbb{C}}_\sigma$.

i) First, assume $\tau = \sigma^-_{(i,j)} \in N^-(\sigma)$ for some $(i,j) \in M(\sigma)$. Pick $\varepsilon \in \mathbb{C}^\times$ and put

$$y_\varepsilon = x_{i,i}(\varepsilon).X^t_\sigma \in \Omega_\sigma.$$

[3] Cf. [Me2, Lemma 3.16].

Then

$$(y_\varepsilon)_{r,s} = \begin{cases} \varepsilon, & \text{if } r = i \text{ and } s = j, \\ (X^t_\sigma)_{r,s} = (X^t_\tau)_{r,s} & \text{otherwise.} \end{cases}$$

Clearly, $y_\varepsilon \to X^t_\tau$ as $\varepsilon \to 0$.

ii) Second, suppose $\tau = c\sigma^{\alpha,\beta}_{(i,j)} \in N^+(\sigma)$ for some $(i,j) \in \mathrm{Supp}(\sigma)$, $(\alpha,\beta) \in C_{i,j}(\sigma)$. Pick $\varepsilon \in \mathbb{C}^\times$ and put

$$y_\varepsilon = x_{\alpha,i}(\varepsilon^{-1}).x_{j,\beta}(-\varepsilon^{-1}).x_{i,i}(\varepsilon).X^t_\sigma \in \Omega_\sigma.$$

Then

$$(y_\varepsilon)_{r,s} = \begin{cases} 1, & \text{if either } r = \alpha \text{ and } s = j, \text{ or } r = i \text{ and } s = \beta, \\ \varepsilon, & \text{if } r = i \text{ and } s = j, \\ (X^t_\sigma)_{r,s} = (X^t_\tau)_{r,s} & \text{otherwise.} \end{cases}$$

Hence $y_\varepsilon \to X^t_\tau$ as $\varepsilon \to 0$.

iii) Third, let $\tau = a\sigma^{(\alpha,\beta)}_{(i,j)} \in N^0(\sigma)$ for some $(i,j) \in \mathrm{Supp}(\sigma)$, $(\alpha,\beta) \in A_{i,j}(\sigma)$. Pick $\varepsilon \in \mathbb{C}^\times$ and put

$$y_\varepsilon = x_{\beta,i}(\varepsilon^{-1}).x_{i,i}(\varepsilon).x_{\alpha,\alpha}(-\varepsilon).X^t_\sigma \in \Omega_\sigma.$$

Then

$$(y_\varepsilon)_{r,s} = \begin{cases} 1, & \text{if either } r = \beta \text{ and } s = j, \text{ or } r = \alpha \text{ and } s = i, \\ \varepsilon \text{ (resp. } -\varepsilon), & \text{if } r = i \text{ and } s = j \text{ (resp. } r = \alpha \text{ and } s = \beta), \\ (X^t_\sigma)_{r,s} = (X^t_\tau)_{r,s} & \text{otherwise.} \end{cases}$$

Thus, $y_\varepsilon \to X^t_\tau$ as $\varepsilon \to 0$.

iv) Now, if $\tau = b\sigma^{(\alpha,\beta)}_{(i,j)} \in N^0(\sigma)$ for some $(i,j) \in \mathrm{Supp}(\sigma)$, $(\alpha,\beta) \in B_{i,j}(\sigma)$, then pick $\varepsilon \in \mathbb{C}^\times$, $\varepsilon \neq 1$ and put

$$y_\varepsilon = x_{j,\beta}(-\varepsilon^{-1}).x_{i,\alpha}(\varepsilon^{-1}).x_{\alpha,\alpha}(\varepsilon).x_{i,i}(\varepsilon - \varepsilon^{-1}).X^t_\sigma \in \Omega_\sigma.$$

Then

$$(y_\varepsilon)_{r,s} = \begin{cases} 1, & \text{if either } r = i \text{ and } s = \beta, \text{ or } r = \alpha \text{ and } s = j, \\ \varepsilon, & \text{if either } r = i \text{ and } s = j, \text{ or } r = \alpha \text{ and } s = \beta, \\ (X^t_\sigma)_{r,s} = (X^t_\tau)_{r,s} & \text{otherwise.} \end{cases}$$

We see that $y_\varepsilon \to X^t_\tau$ as $\varepsilon \to 0$.

v) Finally, suppose $\tau = \sigma^{\rightarrow}_{(i,j)} \in N^0(\sigma)$ (resp. $\tau = \sigma^{\uparrow}_{(i,j)} \in N^0(\sigma)$) for some $(i,j) \in \mathrm{Supp}(\sigma)$. Pick $\varepsilon \in \mathbb{C}^\times$ and put

$$y_\varepsilon = x_{j,m}(-\varepsilon^{-1}).x_{i,i}(\varepsilon).X^t_\sigma \in \Omega_\sigma \text{ (resp. } y_\varepsilon = x_{m,i}(\varepsilon^{-1}).x_{i,i}(\varepsilon).X^t_\sigma \in \Omega_\sigma).$$

Then

$$(y_\varepsilon)_{r,s} = \begin{cases} 1, & \text{if } r = i \text{ and } s = m \text{ (resp. } r = m \text{ and } s = j), \\ \varepsilon, & \text{if } r = i \text{ and } s = j, \\ (X^t_\sigma)_{r,s} = (X^t_\tau)_{r,s} & \text{otherwise.} \end{cases}$$

We conclude that $y_\varepsilon \to X^t_\tau$ as $\varepsilon \to 0$. The result follows. □

Things now are ready to prove the "if" direction of Theorem 1.7 (see the next Section for the proofs of some technical but crucial Lemmas).

Proposition 2.10. *Let σ, τ be involutions in S_n. If $\tau \leq^* \sigma$, then $\Omega_\tau \subseteq \overline{\Omega}_\sigma$.*

PROOF. Denote $s(\sigma) = |\mathrm{Supp}(\sigma)|$ and put $L(\sigma) = L^+(\sigma) \cup L^-(\sigma) \cup L^0(\sigma)$, where[4]

[4]Cf. [Me2, Subsection 3.7].

$$
\begin{aligned}
L^-(\sigma) &= \{\sigma' \in S_n^2 \mid \sigma' \leq^* \sigma,\ s(\sigma') < s(\sigma),\ \text{and if } \sigma' \leq^* w <^* \sigma,\ s(w) < s(\sigma),\ \text{then } w = \sigma'\},\\
L^+(\sigma) &= \{\sigma' \in S_n^2 \mid \sigma' \leq^* \sigma,\ s(\sigma') > s(\sigma),\ \text{and if } \sigma' \leq^* w <^* \sigma,\ \text{then } w = \sigma'\},\\
L^0(\sigma) &= \{\sigma' \in S_n^2 \mid \sigma' \leq^* \sigma,\ s(\sigma') = s(\sigma),\ \text{and if } \sigma' \leq^* w <^* \sigma,\ \text{then } w = \sigma'\}.
\end{aligned}
$$

Evidently, there exist involutions $\sigma = w_1 \geq^* w_2 \geq^* \ldots \geq^* w_k = \tau$ such that $w_{i+1} \in L(w_i)$ for all $1 \leq i < k$, so we can assume $\tau \in L(\sigma)$. But Lemmas 3.5, 3.6 and 3.7 show that $N^-(\sigma) = L^-(\sigma)$, $N^0(\sigma) = L^0(\sigma)$ and $N^+(\sigma) = L^+(\sigma)$ respectively, so $L(\sigma) = \mathrm{Near}(\sigma)$. Applying Proposition 2.9, we conclude the proof. □

2.4. In this Subsection, we prove Theorem 1.10 (using technical Lemmas proved in the next Section). Let σ, $\tau \in S_n^2$. Recall that

$$
\begin{aligned}
&\tau \leq_B \sigma \text{ if and only if } R(\dot{\tau}) \leq R(\dot{\sigma}),\\
&\tau \leq^* \sigma \text{ if and only if } R^*_\tau = R(\dot{\tau})_{\mathrm{low}} \leq R^*_\sigma = R(\dot{\sigma})_{\mathrm{low}},
\end{aligned}
$$

so $\tau \leq_B \sigma$ implies $\tau \leq^* \sigma$ (see Subsection 1.4).

In order to check that the converse holds, denote

$$
\begin{aligned}
L_B(\sigma) &= \{\sigma' \in S_n^2 \mid s' \leq_B \sigma \text{ and if } \sigma' \leq_B w <_B \sigma,\ \text{then } w = \sigma'\},\\
L_*(\sigma) &= \{\sigma' \in S_n^2 \mid s' \leq^* \sigma \text{ and if } \sigma' \leq^* w <^* \sigma,\ \text{then } w = \sigma'\}.
\end{aligned}
\tag{2}
$$

Clearly, $L_*(\sigma) = L'(\sigma) \cup L^+(\sigma) \cup L^0(\sigma)$, where

$$
L'(\sigma) = \{\sigma' \in S_n^2 \mid \sigma' \leq^* \sigma,\ s(\sigma') < s(\sigma),\ \text{and if } \sigma' \leq^* w <^* \sigma,\ \text{then } w = \sigma'\} \subseteq L^-(\sigma).
$$

(In general, $L'(\sigma) \subsetneq L^-(\sigma)$.) Put also

$$
N'(\sigma) = \{\sigma^-_{(i,j)}, (i,j) \in M(\sigma) \text{ and } \sigma(m) \neq m \text{ for all } j \leq m \leq i\} \subseteq N^-(\sigma).
$$

It follows from [In2, Theorem 5.2] that $L_B(\sigma) = \mathrm{Near}'(\sigma) = N'(\sigma) \cup N^+(\sigma) \cup N^0(\sigma)$. But Lemmas 3.6, 3.7 and 3.8 show that $N^0(\sigma) = L^0(\sigma)$, $N^+(\sigma) = L^+(\sigma)$ and $N'(\sigma) = L'(\sigma)$ respectively, so $\mathrm{Near}'(\sigma) = L_*(\sigma)$. Hence the conditions $\sigma \geq^* \tau$ and $\sigma \geq_B \tau$ are equivalent; this proves Theorem 1.10 and so concludes the proof of Theorem 1.1. Furthermore, this gives the following combinatorial description of Bruhat–Chevalley order on S_n^2:

$$
\tau \leq_B \sigma \text{ if and only if } R^*_\tau \leq R^*_\sigma.
$$

3. Proofs of technical Lemmas

3.1. It turns out that the equalities $L(\sigma) = \mathrm{Near}(\sigma)$ and $L_*(\sigma) = \mathrm{Near}'(\sigma)$ play a key role in the proofs of Proposition 2.10 and Theorem 1.10 respectively. The proofs of these equalities are completely straightforward, but rather long. First, we will prove that $\mathrm{Near}(\sigma) \subseteq L(\sigma)$. Obviously,

$$
s(\tau) = s(\sigma) \pm 1 \text{ for all } \tau \in N^\pm(\sigma),\ \text{and } s(\tau) = s(\sigma) \text{ for all } \tau \in N^0(\sigma). \tag{3}
$$

Note that if $\sigma \geq^* w \geq^* \tau$, then

$$
Y \cap \mathrm{Supp}(\sigma) = Y \cap \mathrm{Supp}(\tau) \Longrightarrow Y \cap \mathrm{Supp}(w) = Y \cap \mathrm{Supp}(\sigma) = Y \cap \mathrm{Supp}(\tau) \tag{4}
$$

for all $Y \subseteq \Phi$ such that if $(a,b) \in Y$, $(c,d) \in \Phi$ and $(c,d) > (a,b)$, then $(c,d) \in Y$. Note also that

$$
\sigma = \tau \Longleftrightarrow R^*_\sigma = R^*_\tau, \tag{5}
$$

and, moreover,

$$Y \cap \mathrm{Supp}(\sigma) = Y \cap \mathrm{Supp}(\tau) \iff (R^*_\sigma)_{r,s} = (R^*_\tau)_{r,s} \text{ for all } (r,s) \in Y. \tag{6}$$

Let $1 \leq i, j \leq n$. It's very convenient to put

$$\mathcal{R}_i = \{(i,s) \in \Phi \mid 1 \leq s < i\},\ \mathcal{C}_j = \{(r,j) \in \Phi \mid j < r \leq n\}.$$

Definition 3.1. The sets $\mathcal{R}_i$, $\mathcal{C}_j$ are called the ith *row* and the jth *column* of Φ respectively. Note that if $\sigma \in S_n^2$, then

$$|\mathrm{Supp}(\sigma) \cap (\mathcal{R}_i \cup \mathcal{C}_i)| \leq 1 \text{ for all } 1 \leq i \leq n. \tag{7}$$

Lemma 3.2. *Let $\sigma \in S_n^2$. One has*[5] $N^-(\sigma) \subseteq L^-(\sigma)$.

PROOF. Suppose $\tau = \sigma^-_{(i,j)}$ for some $(i,j) \in M(\sigma)$. By (3), $s(\tau) = s(\sigma) - 1 < s(\sigma)$. Put

$$Y = \Phi \setminus \{(p,q) \in \Phi \mid (p,q) \not\leq (i,j)\}.$$

Then $\mathrm{Supp}(\tau) = \mathrm{Supp}(\tau) \cap Y = \mathrm{Supp}(\sigma) \cap Y$.

Now, assume that there exists $w \in S_n^2$ such that $\tau \leq^* w <^* \sigma$ and $s(w) < s(\sigma)$. Then, by (4), $Y \cap \mathrm{Supp}(\sigma) = Y \cap \mathrm{Supp}(\tau) = Y \cap \mathrm{Supp}(w)$, so $s(w) \geq s(\tau) = s(\sigma) - 1$. Thus, $s(w) = s(\tau)$ and $\mathrm{Supp}(w) = \mathrm{Supp}(\tau)$, so $w = \tau$. $\square$

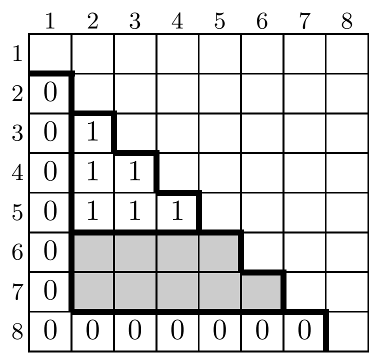
Lemma 3.3. *Let $\sigma \in S_n^2$. One has*[6] $N^0(\sigma) \subseteq L^0(\sigma)$.

PROOF. i) Let $\tau \in N^0(\sigma)$, then, by (3), $s(\tau) = s(\sigma)$. First, suppose $\tau = \sigma^{\uparrow}_{(i,j)}$ for some $(i,j) \in \mathrm{Supp}(\sigma)$ (the case $\tau = \sigma^{\rightarrow}_{(i,j)}$ is completely similar). Let $\mathrm{Supp}(\tau) \setminus \mathrm{Supp}(\sigma) = \{(m,j)\}$. Put $Y = Y_0 \cup Y_1$ and $\widetilde{Y} = \Phi \setminus Y$, where

$$Y_0 = \{(p,q) \in \Phi \mid (p,q) \not\leq (i,j)\},$$
$$Y_1 = \{(p,q) \in \Phi \mid (p,q) \leq (m,j)\}.$$

Then $(R^*_\sigma)_{r,s} = (R^*_\tau)_{r,s}$ for all $(r,s) \in Y$.

For example, let $n = 8$, $i = 7$, $j = 2$, $m = 5$. On the picture below boxes from Y_0 are filled by 0's, boxes from Y_1 are filled by 1's, and boxes from $\widetilde{Y}$ are grey.



Now, assume there exists $w \in S_n^2$ such that $\tau \leq^* w <^* \sigma$. By (5), it's enough to show that $(R^*_w)_{r,s} = (R^*_\tau)_{r,s}$ for all $(r,s) \in \widetilde{Y}$, because $(R^*_\sigma)_{r,s} = (R^*_\tau)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in Y$. Note that $(R^*_\tau)_{r,s} = (R^*_\sigma)_{r,s} - 1$ for all $(r,s) \in \widetilde{Y}$. Further, by definition of $\sigma^{\uparrow}_{(i,j)}$ (see Subsection 2.1), $\mathrm{Supp}(\sigma) \cap \widetilde{Y} = \{(i,j)\}$ and $\mathrm{Supp}(\tau) \cap \widetilde{Y} = \emptyset$.

[5] Cf. [Me2, Lemma 3.8].

[6] Cf. [Me2, Lemmas 3.11–3.14].

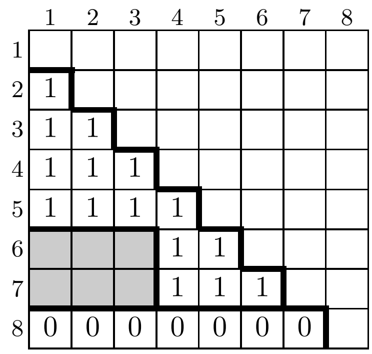
Since $\tau \leq^* w <^* \sigma$, there exists $(k,j) \in \mathrm{Supp}(w)$ such that $m \leq k \leq i$. We claim that $\mathrm{Supp}(w) \cap (\widetilde{Y} \setminus \mathcal{R}_i) = \emptyset$. Indeed, assume the converse holds, i.e., there exists $(p,q) \in \mathrm{Supp}(w) \cap (\widetilde{Y} \setminus \mathcal{R}_i)$. Then $m < p < i$. By definition of $\sigma^{\uparrow}_{(i,j)}$, there are no r such that $m < r < i$ and $\sigma(r) = r$. Hence $\mathrm{Supp}(\sigma) \cap Y_0 \cap (\mathcal{R}_p \cup \mathcal{C}_p) \neq \emptyset$. By (4), $\mathrm{Supp}(\sigma) \cap Y_0 = \mathrm{Supp}(\tau) \cap Y_0 = \mathrm{Supp}(w) \cap Y_0$, so $\mathrm{Supp}(w) \cap Y_0 \cap (\mathcal{R}_p \cup \mathcal{C}_p) \neq \emptyset$. But $(p,q) \in \widetilde{Y}$, hence $|\mathrm{Supp}(\sigma) \cap (\mathcal{R}_p \cup \mathcal{C}_p)| \geq 2$. This contradicts (7). Thus, $\mathrm{Supp}(w) \cap (\widetilde{Y} \setminus \mathcal{R}_i) = \emptyset$, as required.

In particular, either $k = i$ or $k = m$. If $k = i$, then $\mathrm{Supp}(w) \cap \widetilde{Y} = \mathrm{Supp}(\sigma) \cap \widetilde{Y} = \{(i,j)\}$, so $\mathrm{Supp}(w) \cap (\widetilde{Y} \cup Y_0) = \mathrm{Supp}(\sigma) \cap (\widetilde{Y} \cup Y_0)$, and, by (6), $(R^*_\sigma)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in \widetilde{Y} \cup Y_0$. But $(R^*_\sigma)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in Y_1$, hence $R^*_\sigma = R^*_w$, and, by (5), $\sigma = w$, a contradiction. Thus, $k = m$. This means that $\mathrm{Supp}(w) \cap \widetilde{Y} = \mathrm{Supp}(\tau) \cap \widetilde{Y} = \emptyset$, so $\mathrm{Supp}(w) \cap (\widetilde{Y} \cup Y_0) = \mathrm{Supp}(\tau) \cap (\widetilde{Y} \cup Y_0)$, and, by (6), $(R^*_\tau)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in \widetilde{Y} \cup Y_0$. But $(R^*_\tau)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in Y_1$, hence $R^*_\tau = R^*_w$, and, by (5), $\tau = w$, as required.

ii) Second, assume $\tau = b\sigma^{(\alpha,\beta)}_{(i,j)}$ for some $(i,j) \in \mathrm{Supp}(\sigma)$, $(\alpha,\beta) \in B_{i,j}(\sigma)$. Then $(R^*_\sigma)_{r,s} = (R^*_\tau)_{r,s}$ for all $(r,s) \in Y$. Here we put $Y = Y_0 \cup Y_1$ and $\widetilde{Y} = \Phi \setminus Y$, where

$$\begin{aligned} Y_0 &= \{(p,q) \in \Phi \mid (p,q) \not\leq (\alpha,\beta)\}, \\ Y_1 &= \{(p,q) \in \Phi \mid (p,q) \leq (i,\beta) \text{ or } (p,q) \leq (\alpha,j)\}. \end{aligned}$$

For example, let $n = 8$, $i = 5$, $j = 4$, $\alpha = 7$, $\beta = 1$. On the picture below boxes from Y_0 are filled by 0's, boxes from Y_1 are filled by 1's, and boxes from $\widetilde{Y}$ are grey.



Now, assume there exists $w \in S_n^2$ such that $\tau \leq^* w <^* \sigma$. By (5), it's enough to show that $(R^*_w)_{r,s} = (R^*_\tau)_{r,s}$ for all $(r,s) \in \widetilde{Y}$, because $(R^*_\sigma)_{r,s} = (R^*_\tau)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in Y$. Note that $(R^*_\tau)_{r,s} = (R^*_\sigma)_{r,s} - 1$ for all $(r,s) \in \widetilde{Y}$. Further, by definition of $b\sigma^{(\alpha,\beta)}_{(i,j)}$ (see Subsection 2.1), $\mathrm{Supp}(\sigma) \cap \widetilde{Y} = \{(\alpha,\beta)\}$ and $\mathrm{Supp}(\tau) \cap \widetilde{Y} = \emptyset$.

Since $\tau \leq^* w <^* \sigma$, there exists $(k,\beta) \in \mathrm{Supp}(\sigma)$ such that $i \leq k \leq \alpha$. If $k = \alpha$, then $w <^* \sigma$ follows $\mathrm{Supp}(w) \cap \widetilde{Y} = \mathrm{Supp}(\sigma) \cap \widetilde{Y} = \{(\alpha,\beta)\}$, so $\mathrm{Supp}(w) \cap (\widetilde{Y} \cup Y_0) = \mathrm{Supp}(\sigma) \cap (\widetilde{Y} \cup Y_0)$, and, by (6), $(R^*_\sigma)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in \widetilde{Y} \cup Y_0$. But $(R^*_\sigma)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in Y_1$, hence $R^*_\sigma = R^*_w$, and, by (5), $\sigma = w$, a contradiction. Thus, $k < \alpha$.

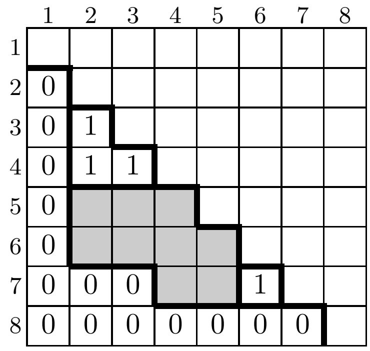
Similarly, there exists $(\alpha,l) \in \mathrm{Supp}(w)$ such that $\beta < l \leq j$. If $k > i$ or $l < j$, then $(R^*_w)_{k,l} > (R^*_\sigma)_{k,l}$, which contradicts $w <^* \sigma$. Hence $k = i$ and $l = j$, i.e., (i,β) and (α,j) belong to $\mathrm{Supp}(w)$. This implies $\mathrm{Supp}(w) \cap \widetilde{Y} = \mathrm{Supp}(\tau) \cap \widetilde{Y} = \emptyset$, because if $(p,q) \in \mathrm{Supp}(w) \cap \widetilde{Y}$, then $(R^*_w)_{p,j} > (R^*_\sigma)_{p,j}$ and $(R^*_w)_{i,q} > (R^*_\sigma)_{i,q}$, a contradiction. Thus, $\mathrm{Supp}(w) \cap (\widetilde{Y} \cup Y_0) = \mathrm{Supp}(\tau) \cap (\widetilde{Y} \cup Y_0)$, and, by (6), $(R^*_\tau)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in \widetilde{Y} \cup Y_0$. But $(R^*_\tau)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in Y_1$, hence $R^*_\tau = R^*_w$, and, by (5), $\tau = w$, as required.

iii) Finally, suppose $\tau = a\sigma^{(\alpha,\beta)}_{(i,j)}$ for some $(i,j) \in \mathrm{Supp}(\sigma)$, $(\alpha,\beta) \in A_{i,j}(\sigma)$. Then $(R^*_\sigma)_{r,s} = (R^*_\tau)_{r,s}$

for all $(r, s) \in Y$. Here we put $Y = Y_0 \cup Y_1$ and $\widetilde{Y} = \Phi \setminus Y$, where

$$Y_0 = \{(p,q) \in \Phi \mid (p,q) \nleq (i,j) \text{ and } (p,q) \nleq (\alpha,\beta)\},$$
$$Y_1 = \{(p,q) \in \Phi \mid (p,q) \leq (\beta,j) \text{ or } (p,q) \leq (\alpha,i)\}.$$

For example, let $n = 8$, $i = 6$, $j = 2$, $\alpha = 7$, $\beta = 4$. On the picture below boxes from Y_0 are filled by 0's, boxes from Y_1 are filled by 1's, and boxes from $\widetilde{Y}$ are grey.



Now, assume there exists $w \in S_n^2$ such that $\tau \leq^* w <^* \sigma$. By (5), it's enough to show that $(R_w^*)_{r,s} = (R_\tau^*)_{r,s}$ for all $(r,s) \in \widetilde{Y}$, because $(R_\sigma^*)_{r,s} = (R_\tau^*)_{r,s} = (R_w^*)_{r,s}$ for all $(r,s) \in Y$. Note that, by definition of $a\sigma_{(i,j)}^{(\alpha,\beta)}$ (see Subsection 2.1), $\mathrm{Supp}(\sigma) \cap \widetilde{Y} = \{(i,j), (\alpha,\beta)\}$ and $\mathrm{Supp}(\tau) \cap \widetilde{Y} = \emptyset$.

Since $\tau \leq^* w <^* \sigma$, there exists $(k, j) \in \mathrm{Supp}(\sigma)$ such that $\beta \leq k \leq i$. We claim that $\mathrm{Supp}(w) \cap Y' = \emptyset$, where $Y' = \widetilde{Y} \setminus \{(r,s) \in \Phi \mid (r,s) \geq (i,\beta)\}$). Indeed, assume the converse holds, i.e., there exists $(p,q) \in \mathrm{Supp}(w) \cap Y'$. Assume $\beta < p < i$ (the case $\beta < q < i$ is similar). By definition of $a\sigma_{(i,j)}^{(\alpha,\beta)}$, there are no r such that $i < r < \beta$ and $\sigma(r) = r$. Hence $\mathrm{Supp}(\sigma) \cap Y_0 \cap (\mathcal{R}_p \cup \mathcal{C}_p) \neq \emptyset$. By (4), $\mathrm{Supp}(\sigma) \cap Y_0 = \mathrm{Supp}(\tau) \cap Y_0 = \mathrm{Supp}(w) \cap Y_0$, so $\mathrm{Supp}(w) \cap Y_0 \cap (\mathcal{R}_p \cup \mathcal{C}_p) \neq \emptyset$. But $(p,q) \in \widetilde{Y}$, hence $|\mathrm{Supp}(\sigma) \cap (\mathcal{R}_p \cup \mathcal{C}_p)| \geq 2$. This contradicts (7). Thus, $\mathrm{Supp}(w) \cap Y' = \emptyset$.

In particular, either $k = \beta$ or $k = i$, i.e., either $(\beta, j) \in \mathrm{Supp}(w)$ or $(i,j) \in \mathrm{Supp}(w)$. Similarly, either $(\alpha,\beta) \in \mathrm{Supp}(w)$ or $(\alpha, i) \in \mathrm{Supp}(w)$. If $(i,j) \in \mathrm{Supp}(w)$, then, by (7), $(\alpha, i) \notin \mathrm{Supp}(w)$, so $(\alpha,\beta) \in \mathrm{Supp}(w)$. Consequently, $\mathrm{Supp}(w) \cap \widetilde{Y} = \mathrm{Supp}(\sigma) \cap \widetilde{Y} = \{(i,j), (\alpha,\beta)\}$, so $\mathrm{Supp}(w) \cap (\widetilde{Y} \cup Y_0) = \mathrm{Supp}(\sigma) \cap (\widetilde{Y} \cup Y_0)$, and, by (6), $(R_\sigma^*)_{r,s} = (R_w^*)_{r,s}$ for all $(r,s) \in \widetilde{Y} \cup Y_0$. But $(R_\sigma^*)_{r,s} = (R_w^*)_{r,s}$ for all $(r,s) \in Y_1$, hence $R_\sigma^* = R_w^*$, and, by (5), $\sigma = w$, a contradiction. Hence $(\beta, j) \in \mathrm{Supp}(w)$. By (7), $(\alpha,\beta) \notin \mathrm{Supp}(w)$, so $(\alpha, i) \in \mathrm{Supp}(w)$. This implies $\mathrm{Supp}(w) \cap \widetilde{Y} = \mathrm{Supp}(\tau) \cap \widetilde{Y} = \emptyset$. Thus, $\mathrm{Supp}(w) \cap (\widetilde{Y} \cup Y_0) = \mathrm{Supp}(\tau) \cap (\widetilde{Y} \cup Y_0)$, and, by (6), $(R_\tau^*)_{r,s} = (R_w^*)_{r,s}$ for all $(r,s) \in \widetilde{Y} \cup Y_0$. But $(R_\tau^*)_{r,s} = (R_w^*)_{r,s}$ for all $(r,s) \in Y_1$, hence $R_\tau^* = R_w^*$, and, by (5), $\tau = w$, as required. $\square$

Lemma 3.4. *Let $\sigma \in S_n^2$. One has*[7] $N^+(\sigma) \subseteq L^+(\sigma)$.

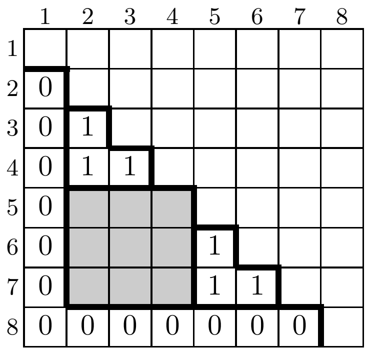
Proof. Let $\tau \in N^+(\sigma)$, then, by (3), $s(\tau) = s(\sigma) + 1 > s(\sigma)$. Suppose $\tau = c\sigma_{(i,j)}^{\alpha,\beta}$ for some $(i,j) \in \mathrm{Supp}(\sigma)$, $(\alpha,\beta) \in C_{i,j}(\sigma)$. Then $(R_\sigma^*)_{r,s} = (R_\tau^*)_{r,s}$ for all $(r,s) \in Y$. Here we put $Y = Y_0 \cup Y_1$ and $\widetilde{Y} = \Phi \setminus Y$, where

$$Y_0 = \{(p,q) \in \Phi \mid (p,q) \nleq (i,j)\},$$
$$Y_1 = \{(p,q) \in \Phi \mid (p,q) \leq (\alpha,j) \text{ or } (p,q) \leq (i,\beta)\}.$$

For example, let $n = 8$, $i = 7$, $j = 2$, $\alpha = 4$, $\beta = 5$. On the picture below boxes from Y_0 are filled

[7] If $\sigma \geq \tau$, then $s(\sigma) \geq s(\tau)$, so there are no analogues to this Lemma in [Me2].

by 0's, boxes from Y_1 are filled by 1's, and boxes from $\widetilde{Y}$ are grey.



Now, assume there exists $w \in S_n^2$ such that $\tau \leq^* w <^* \sigma$. By (5), it's enough to show that $(R^*_w)_{r,s} = (R^*_\tau)_{r,s}$ for all $(r,s) \in \widetilde{Y}$, because $(R^*_\sigma)_{r,s} = (R^*_\tau)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in Y$. Note that $(R^*_\tau)_{r,s} = (R^*_\sigma)_{r,s} - 1$ for all $(r,s) \in \widetilde{Y}$. Further, by definition of $c\sigma^{\alpha,\beta}_{(i,j)}$ (see Subsection 2.1), $\mathrm{Supp}(\sigma) \cap \widetilde{Y} = \{(i,j)\}$ and $\mathrm{Supp}(\tau) \cap \widetilde{Y} = \emptyset$.

Since $\tau \leq^* w <^* \sigma$, there exists $(k,j) \in \mathrm{Supp}(\sigma)$ such that $\alpha \leq k \leq i$. If $k = i$, then $w < \sigma$ follows $\mathrm{Supp}(w) \cap \widetilde{Y} = \mathrm{Supp}(\sigma) \cap \widetilde{Y} = \{(i,j)\}$, so

$$\mathrm{Supp}(w) \cap (\widetilde{Y} \cup Y_0) = \mathrm{Supp}(\sigma) \cap (\widetilde{Y} \cup Y_0)$$

and, by (6), $(R^*_\sigma)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in \widetilde{Y} \cup Y_0$. But $(R^*_\sigma)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in Y_1$, hence $R^*_\sigma = R^*_w$, and, by (5), $\sigma = w$, a contradiction. Thus, $k < i$.

Similarly, there exists $(i,l) \in \mathrm{Supp}(w)$ such that $j < l < \beta$. We claim that either $l \leq \alpha$ or $l = \beta$. Indeed, assume the contrary, i.e., $\beta > l > \alpha$. By definition of $c\sigma^{\alpha,\beta}_{(i,j)}$, there are no s such that $\alpha < s < \beta$ and $\sigma(r) = r$. Hence $\mathrm{Supp}(\sigma) \cap Y_0 \cap (\mathcal{R}_l \cup \mathcal{C}_l) \neq \emptyset$. By (4), $\mathrm{Supp}(\sigma) \cap Y_0 = \mathrm{Supp}(\tau) \cap Y_0 = \mathrm{Supp}(w) \cap Y_0$, so $\mathrm{Supp}(w) \cap Y_0 \cap (\mathcal{R}_l \cup \mathcal{C}_l) \neq \emptyset$. But $(i,l) \in \widetilde{Y}$, hence $|\mathrm{Supp}(\sigma) \cap (\mathcal{R}_l \cup \mathcal{C}_l)| \geq 2$. This contradicts (7). Thus, either $l \leq \alpha$ or $l = \beta$.

If $l = \alpha$, then $k = \alpha$ contradicts (7), so $k \neq \alpha$, i.e., $k > \alpha$. In this case, $(R^*_w)_{k,l} > (R^*_\sigma)_{k,l}$, which contradicts $\sigma >^* w$. Hence either $l < \alpha$ or $l = \beta$. If $l < \alpha$, then $(R^*_w)_{k,l} > (R^*_\sigma)_{k,l}$, as above. This contradicts $\sigma >^* w$, so $l = \beta$. Similarly, $k = \alpha$, Thus, $\mathrm{Supp}(w) \cap (\widetilde{Y} \cup Y_0) = \mathrm{Supp}(\tau) \cap (\widetilde{Y} \cup Y_0)$, and, by (6), $(R^*_\tau)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in \widetilde{Y} \cup Y_0$. But $(R^*_\tau)_{r,s} = (R^*_w)_{r,s}$ for all $(r,s) \in Y_1$, hence $R^*_\tau = R^*_w$, and, by (5), $\tau = w$, as required. $\square$

3.2. In this Subsection, we will prove the most complicated parts of Proposition 2.10. Namely, we will show that $L(\sigma) \subseteq \mathrm{Near}(\sigma)$ for all $\sigma \in S_n^2$, and so these sets coincide. Note that

$$w_0 = (n,1)(n-1,2)\ldots(n-n_0+1,n_0),$$

where $n_0 = [n/2]$, is the maximal element of S_n^2 with respect to the partial order $\leq^*$ on S_n^2.

Lemma 3.5. *Let $\sigma \in S_n^2$. Then $L^-(\sigma) = N^-(\sigma)$.*

Proof. By Lemma 3.2, it's enough to check that $L^-(\sigma) \subseteq N^-(\sigma)$. We must show that

$$\begin{array}{l}\text{if } \tau \leq^* \sigma \text{ and } s(\tau) < s(\sigma),\\ \text{then there exists } \sigma' \in N^-(\sigma) \text{ such that } \tau \leq^* \sigma' <^* \sigma.\end{array} \qquad (8)$$

We will proceed by induction on n (for $n = 1$, there is nothing to prove). The proof is rather long, so we split it into six steps.

i) Let $\sigma = w_0$, the maximal element of S_n^2 with respect to $\leq^*$, $s(\tau) < s(w_0)$ and $\tau <^* w_0$. Let σ' be the involution such that $\mathrm{Supp}(\sigma') = \mathrm{Supp}(\sigma) \setminus \{(n_0, n-n_0+1)\}$, where $n_0 = [n/2]$. Since $(n_0, n-n_0+1) \in M(w_0)$, $\sigma' \in N^-(w_0)$. But $s(\tau) < s(w_0)$ implies

$$(R_\tau^*)_{i,j} \leq |\mathrm{Supp}(\tau)| \leq |\mathrm{Supp}(w_0)| - 1 \leq (R_{\sigma'}^*)_{i,j}$$

for all $(i,j) \in \Phi$, so $\sigma' \geq^* \tau$. Therefore, we may also use the second (downward) induction on $\leq^*$.

ii) Let $\sigma = (i_1, j_1)\dots(i_s, j_s) \in S_n^2$, $\sigma <^* w_0$, $\tau = (p_1, q_1)\dots(p_t, q_t) <^* \sigma$ and $s > t$. Consider the following conditions:

$$\begin{aligned}
&\text{a) There exists } k \leq n_0 \text{ such that } j_l = l \text{ for all } 1 \leq l \leq k \text{ and either } i_k = k+1 \text{ or } k = s.\\
&\text{b) There exists } d \leq k \text{ such that } q_l = l \text{ for all } 1 \leq l \leq d \text{ and either } q_{d+1} > k \text{ or } d = t.\\
&\text{c) } (i_l, j_l) = (i_l, l) > (i_{l+1}, j_{l+1}) = (i_{l+1}, l+1), \text{ i.e., } i_l > i_{l+1}, \text{ for all } 1 \leq l \leq k-1.\\
&\text{d) } (p_l, q_l) = (p_l, l) > (p_{l+1}, q_{l+1}) = (p_{l+1}, l+1), \text{ i.e., } p_l > p_{l+1}, \text{ for all } 1 \leq l \leq d-1.\\
&\text{e) } (i_l, j_l) = (i_l, l) \geq (p_l, q_l) = (p_l, l), \text{ i.e., } i_l \geq p_l, \text{ for all } 1 \leq l \leq d.
\end{aligned} \tag{9}$$

Pick $r \leq s$. Define σ_r and τ_r by putting $\mathrm{Supp}(\sigma_r) = \mathrm{Supp}(\sigma) \cap \widetilde{\mathcal{C}}_r$, $\mathrm{Supp}(\tau_r) = \mathrm{Supp}(\tau) \cap \widetilde{\mathcal{C}}_r$, where $\widetilde{\mathcal{C}}_r = \bigcup_{l \leq r} \mathcal{C}_l$. We claim that

$$\begin{aligned}
&\text{if (8) holds for all } \sigma, \tau \text{ satisfying (9),}\\
&\text{then (8) holds for all } \sigma, \tau \in S_n^2.
\end{aligned} \tag{10}$$

Clearly, it's enough to prove that if (8) holds for all σ, τ satisfying (9), and σ_r, τ_r don't satisfy (9) for some $1 \leq r \leq s$, then (8) holds for σ, τ. We will proceed by induction on r. Evidently, there exist $w_1 = \sigma$, w_2, …, $w_h = \tau \in S_n^2$ such that $w_1 >^* w_2 >^* \dots >^* w_h$ and $w_{i+1} \in L^-(w_i)$ for all $1 \leq i < h$, so we may assume $\tau \in L^-(\sigma)$.

The base $r = 1$ is clear. Indeed, if $j_1 > i_1$, then it follows from $\tau <^* \sigma$ that $q_1 > 1$, so σ, τ belong to $\widetilde{S}_{n-1} = \{w \in S_n \mid w(1) = 1\} \cong S_{n-1}$, and we can use the first inductive assumption. Hence σ_1 satisfies (9a); τ_1 satisfies (9b) automatically. In the case $r = 1$ conditions (9c) and (9d) are trivial, so it remains to check (9e). But if $q_1 = 1$ and $p_1 > i_1$, then $\tau \not<^* \sigma$, a contradiction. Thus, σ_1, τ_1 satisfy (9e), as required.

iii) Now, suppose $1 \leq r \leq s$ and σ_r, τ_r satisfy (9). To perform the induction step, we must prove that either σ_{r+1}, τ_{r+1} satisfy (9), too, or (8) holds for σ, τ. This is trivially true if $i_k = k+1$ for some $k \leq r$, so we may assume that $i_l > l+1$ for all $1 \leq l \leq r$. Since $r \leq s$ and σ_r satisfies (9a), $j_r = r$. First, consider the case when $p_l > l+1$ for all $l \leq r_0 = \min\{r, s(\tau_r)\}$. Suppose $j_{r+1} > r+1$, i.e., $\mathrm{Supp}(\sigma) \cap \mathcal{C}_{r+1} = \emptyset$. Put

$$\begin{aligned}
\widetilde{\sigma} &= P_r(\sigma) = (i_1, 2)(i_2, 3)\dots(i_r, r+1)(i_{r+1}, j_{r+1})\dots(i_s, j_s),\\
\widetilde{\tau} &= P_r(\tau) = (p_1, 2)(p_2, 3)\dots(p_{r_0}, r_0+1)(p_{r_0+1}, q_{r_0+1})\dots(p_t, q_t).
\end{aligned}$$

Here the map $P_r \colon S_n^2 \to S_n$ is defined by the following rule: if $\eta = (a_1, b_1)\dots(a_m, b_m) \in S_n^2$, then

$$P_r(\eta) = (a_1, b_1+1)\dots(a_r, b_z+1)(a_{r+1}, b_{z+1})\dots(a_m, b_m),$$

where $b_z \leq r < b_{z+1}$. Note that, in general, $P_r(\eta)$ is *not* an involution.

Clearly, $\widetilde{\sigma}$ and $\widetilde{\tau}$ are *involutions* in S_n. Indeed, it suffice to check that (7) holds for $\mathrm{Supp}(\widetilde{\tau})$ (for $\mathrm{Supp}(\widetilde{\sigma})$, there is nothing to check, because $r+1 < j_{r+1}$). But if $r_0 < r$, then $r_0 + 1 < r+1 \leq q_{r_0+1}$, because τ_r satisfies (9b), so (7) holds for $\mathrm{Supp}(\widetilde{\tau})$. On the other hand, if $r = r_0$, then $\tau <^* \sigma$ yields $\mathrm{Supp}(\tau) \cap \mathcal{C}_{r+1} = \emptyset$ (if the converse holds, then $(R_\sigma^*)_{r+2,r+1} = r < r+1 = (R_\tau^*)_{r+2,r+1}$, a contradiction). Hence $r_0 + 1 < q_{r_0+1}$ and (7) holds for $\mathrm{Supp}(\widetilde{\tau})$. Thus, $\widetilde{\sigma}, \widetilde{\tau} \in S_n^2$.

Further, they belong to $\widetilde{S}_{n-1}$ and $s(\widetilde{\tau}) = t < s = s(\widetilde{\sigma})$, so, by the first induction hypothesis, there exists $\widetilde{w} \in \widetilde{S}_{n-1}$ such that $\widetilde{\sigma} >^* \widetilde{w} \geq^* \widetilde{\tau}$ and $\widetilde{w} = \widetilde{\sigma}^-_{(i,j)} \in N^-(\widetilde{\sigma})$ for some $(i,j) \in M(\widetilde{\sigma})$. Since σ_r satisfies (9a), $M(\widetilde{\sigma}) \cap \mathcal{C}_{l+1} = \emptyset$ for all $1 \leq l \leq r-1$, so $j \geq r+1$. If $j > r+1$, then $(i,j) \in M(\sigma)$ and $w = \sigma^-_{(i,j)} \geq^* \tau$. On the other hand, if $j = r+1$, then $(i,r) \in M(\sigma)$ and $w = \sigma^-_{i,r} \geq^* \tau$. In both cases, $w \in N^-(\sigma)$ and $w \geq^* \tau$, as required.

iv) Then, suppose σ_r, τ_r satisfy (9), $i_l > l+1$ for all $1 \leq l \leq r$, $p_l > l+1$ for all $l \leq r_0 = \min\{r, s(\tau_r)\}$, but $j_{r+1} = r+1$, i.e., $\mathrm{Supp}(\sigma) \cap \mathcal{C}_{r+1} \neq \emptyset$. If $i_{r+1} > i_r$, then put

$$\sigma_0 = (i_1, 1) \ldots (i_{r-1}, r-1)(i_{r+1}, r)(i_r, r+1)(i_{r+2}, j_{r+2}) \ldots (i_s, j_s).$$

Then $\sigma_0 >^* \sigma >^* \tau$ and $s(\sigma_0) = s > t = s(\tau)$, so, by the second inductive assumption, there exists $w_1 \in S_n$ such that $w_1 \geq^* \tau$ and $w_1 = (\sigma_0)^-_{(i,j)} \in N^-(\sigma_0)$ for some $(i,j) \in M(\sigma_0)$. Since σ_r satisfies (9a), $M(\sigma_0) \cap \mathcal{C}_l = \emptyset$ for all $1 \leq l \leq r-1$, so $j \geq r+1$. If $j > r+1$, then $(i,j) \in M(\sigma)$ and $w = \sigma^-_{(i,j)} \geq^* \tau$.

On the other hand, assume $j = r+1$. If $\mathrm{Supp}(\tau) \cap \mathcal{C}_r = \emptyset$, i.e., $s(\tau_r) < r$, then $w = \sigma^-_{(i,r)} \geq^* \tau$. If $\mathrm{Supp}(\tau) \cap \mathcal{C}_r \neq \emptyset$, i.e., $q_r = r$, then set σ_1, τ_1 to be the involutions such that

$$\mathrm{Supp}(\sigma_1) = \mathrm{Supp}(\sigma) \setminus \mathcal{C}_r,\ \mathrm{Supp}(\tau_1) = \mathrm{Supp}(\tau) \setminus \mathcal{C}_r.$$

Put also $\widetilde{\sigma}_1 = P_{r-1}(\sigma_1)$ and $\widetilde{\tau}_1 = P_{r-1}(\tau_1)$. We see that $\widetilde{\sigma}_1, \widetilde{\tau}_1 \in S_n^2$ and $\widetilde{\sigma}_1, \widetilde{\tau}_1 \in \widetilde{S}_{n-1}$. Moreover, $s(\widetilde{\sigma}_1) = s-1 > t-1 = s(\widetilde{\tau}_1)$ and $\widetilde{\sigma}_1 >^* \widetilde{\tau}_1$ (if $\widetilde{\sigma}_1 = \widetilde{\tau}_1$, then $s = t$, a contradiction). Hence, by the first induction hypothesis, there exists $\widetilde{w}_1 \in S_n^2$ such that $\widetilde{w}_1 \geq^* \widetilde{\tau}_1$ and $\widetilde{w}_1 = (\widetilde{\sigma}_1)^-_{(a,b)} \in N^-(\widetilde{\sigma}_1)$ for some $(a,b) \in M(\widetilde{\sigma}_1)$. Since $j_r = r$, $q_r = r$ and σ_r, τ_r satisfy (9), $M(\widetilde{\sigma}_1) \cap \mathcal{C}_{l+1} = \emptyset$ for all $1 \leq l \leq r-1$, so $b \geq r+1$. Hence $(a, b-1) \in M(\sigma)$ and $w = \sigma^-_{(a,b-1)} \geq^* \tau$. We conclude that if σ_{r+1} doesn't satisfy (9c), then (8) holds for σ, τ.

Next, suppose $i_{r+1} < i_r$, but $r_0 = r$, $q_{r+1} = r+1$ and $p_{r+1} > i_{r+1}$. In this case, put

$$\tau_0 = (p_1, 1) \ldots (p_{r-1}, r-1)(p_{r+1}, r)(p_r, r+1)(p_{r_2}, q_{r+2}) \ldots (p_t, q_t).$$

Then $\sigma >^* \tau_0 >^* \tau$, so $\tau \notin L^-(\sigma)$. The cases $i_{r+1} < i_r$, $r_0 = r$, $q_{r+1} = r+1$, $p_r < p_{r+1} < i_{r+1}$, and $i_{r+1} < i_r$, $r_0 < r$, $\mathrm{Supp}(\tau) \cap \mathcal{C}_{r+1} \neq \emptyset$ are similar. Namely, if $i_{r+1} < i_r$, $r_0 = r$, $q_{r+1} = r+1$, $p_r < p_{r+1} < i_{r+1}$, then we define τ_0 as above, and if $i_{r+1} < i_r$, $r_0 < r$, $\mathrm{Supp}(\tau) \cap \mathcal{C}_{r+1} = \{(p_{r_0+1}, r+1)\} \neq \emptyset$, then we put

$$\tau_0 = (p_1, 1) \ldots (p_{r_0}, r_0)(p_{r_0+1}, r)(p_{r_0+2}, q_{r_0+2}) \ldots (p_t, q_t).$$

In both cases, $\sigma >^* \tau_0 >^* \tau$, so $\tau \notin L^-(\sigma)$. We conclude that if τ_{r+1} doesn't satisfy (9d), or σ_{r+1}, τ_{r+1} don't satisfy (9e), then (8) holds for σ, τ.

v) To prove (10), it remains to consider the case when σ_r, τ_r satisfy (9), $i_l > l+1$ for all $1 \leq l \leq r$, but $p_d = d+1$ for some $d \leq r$. (And, consequently, $r_0 = s(\tau_r) = d$.) Suppose $j_{r+1} > r+1$, i.e., $\mathrm{Supp}(\sigma) \cap \mathcal{C}_{r+1} = \emptyset$. Let τ_0 be the involution such that $\mathrm{Supp}(\tau_0) = \mathrm{Supp}(\tau) \setminus \{(d+1, d)\}$; put also $\widetilde{\sigma} = P_r(\sigma)$, $\widetilde{\tau}_0 = P_r(\tau_0)$. Then $\widetilde{\sigma}$, $\widetilde{\tau}_0 \in \widetilde{S}_{n-1} \cap S_n^2$, $\widetilde{\sigma} >^* \widetilde{\tau}_0$ and $s(\widetilde{\sigma}) = s > t > t-1 = s(\widetilde{\tau}_0)$. Hence, by the first inductive assumption, there exists $\widetilde{w} = \widetilde{\sigma}^-_{(i,j)} \in N^-(\widetilde{\sigma})$ such that $\widetilde{w} \geq^* \widetilde{\tau}_0$. Since σ_r satisfies (9a), $M(\widetilde{\sigma}) \cap \mathcal{C}_{l+1} = \emptyset$ for all $1 \leq l \leq r-1$, so $j \geq r+1$. If $j > r+1$, then $(i,j) \in M(\sigma)$ and $w = \sigma^-_{(i,j)} \geq^* \tau$. Similarly, if $j = r+1$ and $r > d$, then $(i_r, r) \in M(\sigma)$ and $w = \sigma^-_{(i_r,r)} \leq^* \tau$.

On the other hand, if $j = r+1$ and $r = d$, then put $\widetilde{\sigma}_1 = P_d(\sigma_1)$, where σ_1 is the involution such that $\mathrm{Supp}(\sigma_1) = \mathrm{Supp}(\sigma) \setminus \{(i_d, d)\}$. In this case, $\widetilde{\sigma}_1$, $\widetilde{\tau}_0 \in \widetilde{S}_{n-1} \cap S_n^2$, $s(\widetilde{\sigma}_1) = s-1 > t-1 = s(\widetilde{\tau}_0)$ and $\widetilde{\sigma}_1 >^* \widetilde{\tau}_0$. Whence the first induction hypothesis shows that there exists $\widetilde{w}_1 = (\sigma_1)^-_{(\alpha,\beta)} \in N^-(\widetilde{\sigma}_1)$ such that $\widetilde{w}_1 \geq^* \widetilde{\tau}_0$. Since σ_r, τ_r satisfy (9), $\beta > r = d$ (and so $\beta > r+1$). Thus, $(\alpha, \beta) \in M(\sigma)$ and $w = \sigma^-_{(\alpha,\beta)} \geq^* \tau$.

Now, suppose $j_{r+1} = r+1$, but $i_{r+1} > i_r$. Arguing as on the previous step, we conclude that there exists $w \in N^-(\sigma)$ such that $w \geq^* \tau$. Thus, if σ_{r+1} doesn't satisfy (9a), then (8) holds for σ, τ.

Next, suppose $j_{r+1} = r+1$, $i_{r+1} < i_r$, but τ_{r+1} doesn't satisfy (9c) or (9d), i.e., $\mathrm{Supp}(\tau) \cap \mathcal{C}_{r+1} = \{(p, r+1)\} \neq \emptyset$ (and, consequently, $r > d$, since (7) holds for τ). If $r > d+1$, then put

$$\tau_2 = (p_1, 1) \dots (p_{d-1}, d-1)(d+1, d)(p, r) \dots (p_t, q_t),$$

i.e., $\mathrm{Supp}(\tau_2) = (\mathrm{Supp}(\tau) \setminus \{(p, r+1)\}) \cup \{(p, r)\}$. Then $\sigma >^* \tau_2 >^* \tau$. Indeed, the last inequality is evident, and the first one follows from $\sigma >^* \tau$ and the fact that σ_r, τ_r satisfy (9). Thus, $\tau \notin L^-(\sigma)$. Similarly, if $r = d+1$, then put

$$\tau_3 = (p_1, 1) \dots (p_{d-1}, d-1)(p, d)(p_{d+2}, q_{d+2}) \dots (p_t, q_t),$$

i.e., $\mathrm{Supp}(\tau_3) = (\mathrm{Supp}(\tau) \setminus \{(d+1, d), (p, r+1)\}) \cup \{(p, d)\}$. In this case, $\sigma >^* \tau_3 >^* \tau$, so $\tau \notin L^-(\sigma)$. The proof of (10) is complete.

vi) Now, we may assume without loss of generality that σ, τ satisfy (9). If $i_l > l+1$ for all $1 \leq l \leq k$, then, by (9a) and (9b),

$$k = s = s(\sigma) \geq d = t = s(\tau).$$

If $q_d > d+1$, then put $\widetilde{\sigma} = P_k(\sigma)$, $\widetilde{\tau} = P_k(\tau) = P_d(\tau)$. As above, $\widetilde{\sigma}, \widetilde{\tau} \in \widetilde{S}_{n-1} \cap S_n^2$, so, by the first induction hypothesis, there exists $\widetilde{w} = \widetilde{\sigma}^-_{(i,j)}$ such that $\widetilde{w} \geq^* \widetilde{\tau}$. Since σ, τ satisfy (9), $M(\widetilde{\sigma}) \cap \mathcal{C}_{l+1} = \emptyset$ for all $1 \leq l \leq k$, so $(i,j) \in M(\sigma)$ and $\tau \leq^* w = \sigma^-_{(i,j)} \in N^-(\sigma)$. On the other hand, if $q_d = d+1$ for some $d \leq k$, then, arguing as on the previous step, we conclude that such an involution w exists.

Finally, assume $i_k = k+1$ for some $k \leq s$. Clearly, $(k+1, k) \in M(\sigma)$. If $\mathrm{Supp}(\tau) \cap \mathcal{C}_k = \emptyset$, i.e., $d < k$, then $\tau \leq^* w = \sigma^-_{(k+1,k)} \in N^-(\sigma)$. At the contrary, if $d = k$, i.e., $(k+1, k) \in \mathrm{Supp}(\tau)$, then we derive an existence of w as on the previous step. The proof is complete. $\square$

Lemma 3.6. *Let $\sigma \in S_n^2$. Then $L^0(\sigma) = N^0(\sigma)$.*

Proof. By Lemma 3.3, it's enough to check that $L^0(\sigma) \subseteq N^0(\sigma)$. By definition,

$$\begin{aligned} L^0(\sigma) &= \{\sigma' \in S_n^2 \mid \sigma' \leq^* \sigma,\ s(\sigma') = s(\sigma), \text{ and if } \sigma' \leq^* w <^* \sigma, \text{ then } w = \sigma'\} \\ &\subseteq \widetilde{L}^0(\sigma) = \{\sigma' \in S_n^2 \mid \sigma' \leq^* \sigma,\ s(\sigma') = s(\sigma), \text{ and if } \sigma' <^* w <^* \sigma, \text{ then } s(w) > s(\sigma)\}, \end{aligned}$$

so it suffice to show that $\widetilde{L}^0(\sigma) \subseteq N^0(\sigma)$, i.e.,

$$\begin{aligned} &\text{if } \tau \leq^* \sigma \text{ and } s(\tau) = s(\sigma), \\ &\text{then there exists } \sigma' \in N^0(\sigma) \cup N^-(\sigma) \text{ such that } \tau \leq^* \sigma' <^* \sigma. \end{aligned} \tag{11}$$

We will proceed by induction on n (for $n = 1$, there is nothing to prove). The proof is rather long, so we split it into seven steps.

i) Let $\sigma = w_0$, the maximal element of S_n^2 with respect to $\leq^*$, $s(\tau) = s(w_0)$ and $\tau < w_0$. Let $n_0 = [n/2]$ and $h = \min\{s \mid 1 \leq s \leq n_0 \text{ and } (n-s+1, s) \notin \mathrm{Supp}(\tau)\}$. (If h doesn't exist, then $R^*_\sigma = R^*_\tau$ and, by (5), $\tau = \sigma$, a contradiction.) If $s < n_0$, then $(R^*_\tau)_{n-s+1,s} \leq s-1$, hence $\tau \leq^* w = b(w_0)^{(n-s+1,s)}_{(n-s,s+1)} \in N^0(w_0)$ (clearly, $(n-s+1, s) \in B_{n-s,s+1}(w_0)$), because

$$(R^*_w)_{i,j} = \begin{cases} s-1, & \text{if } i = n-s+1 \text{ and } j = s, \\ (R^*_{w_0})_{i,j} & \text{otherwise.} \end{cases}$$

On the other hand, if $s = n_0$, then n is odd and either

$$\begin{aligned} &\mathrm{Supp}(\tau) = (\mathrm{Supp}(w_0) \setminus \{(n_0+2, n_0)\}) \cup \{(n_0+1, n_0)\} \text{ or} \\ &\mathrm{Supp}(\tau) = (\mathrm{Supp}(w_0) \setminus \{(n_0+2, n_0)\}) \cup \{(n_0+2, n_0+1)\}. \end{aligned}$$

If the first case occurs, then we put $w = (w_0)^{\uparrow}_{(n_0+2,n_0)}$, and if the second case occurs, we put $w = (w_0)^{\rightarrow}_{(n_0+2,n_0)}$. In both cases, $w \in N^0(w_0)$ and $w \geq^* \tau$ (in fact, $w = \tau$). Therefore, we may also use the second (downward) induction on $\leq^*$.

ii) Let $\sigma = (i_1, j_1) \ldots (i_t, j_t) \in S_n^2$, $\sigma <^* w_0$, $\tau = (p_1, q_1) \ldots (p_t, q_t) <^* \sigma$. Consider the following conditions (cf. (9)):

$$\begin{aligned}
&\text{a) There exists } k \leq t \text{ such that } i_l > l+1 \text{ for all } 1 \leq l \leq k-1.\\
&\text{b) There exists } d \leq k \text{ such that } q_l = l \text{ for all } 1 \leq l \leq d\\
&\quad\ \text{and either } d < k,\ q_{d+1} > k,\ \text{or } d = k,\ p_d = d+1.\\
&\text{c) } j_l = l \text{ for all } 1 \leq l \leq d, \text{ and if } d < k, \text{ then either } i_k = j_k + 1 \text{ or } k = t.\\
&\text{d) } (p_l, q_l) = (p_l, l) > (p_{l+1}, q_{l+1}) = (p_{l+1}, l+1), \text{ i.e., } p_l > p_{l+1}, \text{ for all } 1 \leq l \leq d-1.\\
&\text{e) } (i_l, j_l) = (i_l, l) \geq (p_l, q_l) = (p_l, l), \text{ i.e., } i_l \geq p_l, \text{ for all } 1 \leq l \leq d.
\end{aligned} \tag{12}$$

Pick $r \leq t$. Define σ_r and τ_r as in the previous Lemma. We claim that

$$\begin{aligned}
&\text{if (11) holds for all } \sigma, \tau \text{ satisfying (12),}\\
&\text{then (11) holds for all } \sigma, \tau \in S_n^2.
\end{aligned} \tag{13}$$

Clearly, it's enough to prove that if (11) holds for all σ, τ satisfying (12), and σ_r, τ_r don't satisfy (12) for some $1 \leq r \leq t$, then (11) holds for σ, τ.

We will proceed by induction on r. Evidently, there exist $w_1 = \sigma$, w_2, …, $w_z = \tau \in S_n^2$ such that $w_1 >^* w_2 >^* \ldots >^* w_z$ and $w_{i+1} \in \widetilde{L}^0(w_i)$ for all $1 \leq i < z$, so we may assume $\tau \in \widetilde{L}^0(\sigma)$. The base $r = 1$ is evident (see step ii) of the proof of Lemma 3.5).

iii) Suppose $1 \leq r \leq t$ and σ_r, τ_r satisfy (12). To perform the induction step, we must prove that either σ_{r+1}, τ_{r+1} satisfy (12), too, or (11) holds for σ, τ. This is trivially true if $i_k = j_k + 1$ for some $k \leq r$ or $p_d = d + 1$ for some $d \leq r$, so we may assume that $i_l > j_l + 1$ for all $1 \leq l \leq r$ and $p_l > l + 1$ for all $l \leq r_0 = \min\{r, s(\tau_r)\}$.

Suppose $\mathrm{Supp}(\sigma) \cap \mathcal{C}_{r+1} = \emptyset$. If $\mathrm{Supp}(\tau) \cap \mathcal{C}_{r+1} = \emptyset$, too, then σ_{r+1}, τ_{r+1} satisfy (12). At the contrary, assume $\mathrm{Supp}(\tau) \cap \mathcal{C}_{r+1} = \{(p, r+1)\} \neq \emptyset$. We see that $r_0 = s(\tau_r) = d < r$ (if $r_0 = r$, then $(R^*_\tau)_{r+1,r} = r + 1 > r = (R^*_\sigma)_{r+1,r}$, so $\sigma \not>^* \tau$, a contradiction). If $\mathrm{Supp}(\tau) \cap \mathcal{R}_r = \emptyset$, then put

$$\tau_0 = (p_1, 1) \ldots (p_d, d)(p, r)(p_{d+2}, q_{d+2}) \ldots (p_t, q_t),$$

i.e., $\mathrm{Supp}(\tau_0) = (\mathrm{Supp}(\tau) \setminus \{(p, r+1)\}) \cup \{(p, r)\}$. In this case, $\tau_0 \in S_n^2$ and $\tau <^* \tau_0 \leq^* \sigma$. If $\tau_0 = \sigma$, then $\tau = \sigma^{\rightarrow}_{(p,r)} \in N^0(\sigma)$. At the same time, if $\tau_0 <^* \sigma$, then $\tau \notin \widetilde{L}^0(\sigma)$.

On the other hand, assume $\mathrm{Supp}(\tau) \cap \mathcal{R}_r = \{(r, j)\} \neq \emptyset$. If $i_j > r$, then put τ_0 to be the involution such that

$$\mathrm{Supp}(\tau_0) = (\mathrm{Supp}(\tau) \setminus \{(r, j), (p, r+1)\}) \cup \{(r+1, j), (p, r)\}.$$

Evidently, $\tau <^* \tau_0 \leq^* \sigma$. If $\tau_0 = \sigma$, then $\tau = a\sigma^{(p,r+1)}_{(r,j)} \in N^0(\sigma)$, and if $\tau_0 <^* \sigma$, then $\tau \notin \widetilde{L}^0(\sigma)$.

Next, assume $\mathrm{Supp}(\tau) \cap \mathcal{R}_r = \{(r, j)\} \neq \emptyset$, but $i_j = r$ (in particular, $\mathrm{Supp}(\sigma) \cap \mathcal{C}_r = \emptyset$). It follows from (12d) that $\mathrm{Supp}(\tau) \cap \mathcal{C}_l \cap \mathcal{R}_i = \emptyset$ for all $j < l < r$, $r < i \leq n$. Let $q = [(r - j + 1)/2]$ and

$$\begin{aligned}
z = \max\{s \mid r - q + 1 \leq s < r \text{ and either } &\mathrm{Supp}(\tau) \cap \mathcal{R}_s = \emptyset\\
\text{or } &\mathrm{Supp}(\tau) \cap \mathcal{R}_s = \{(s, h)\}, \text{ where } i_h > s\}.
\end{aligned}$$

It's easy to see that z exists. Indeed, if $\mathrm{Supp}(\sigma) \cap \mathcal{R}_s = \mathrm{Supp}(\tau) \cap \mathcal{R}_s \neq \emptyset$ for all $r - q + 1 \leq s < r$, then $\mathrm{Supp}(\sigma) \cap \mathcal{R}_{r-q+1} = \mathrm{Supp}(\tau) \cap \mathcal{R}_{r-q+1} = \{(r - q + 1, r - q)\}$, which contradicts our assumption. Further,

$\mathrm{Supp}(\tau)\cap\mathcal{C}_z=\emptyset$, because $z\geq r-q+1>q\geq d$. If $\mathrm{Supp}(\tau)\cap\mathcal{R}_z=\emptyset$ (resp. $\mathrm{Supp}(\tau)\cap\mathcal{R}_z=\{(z,h)\}$), then put τ_0 to be the involution such that

$$\begin{aligned}&\mathrm{Supp}(\tau_0)=(\mathrm{Supp}(\tau)\setminus\{(p,r+1)\})\cup\{(p,z)\} \text{ (resp.}\\&\mathrm{Supp}(\tau_0)=(\mathrm{Supp}(\tau)\setminus\{(z,h),(p,r+1)\})\cup\{(r+1,h),(p,z)\}).\end{aligned}$$

Clearly, $\tau<^*\tau_0\leq^*\sigma$. If $\tau_0<^*\sigma$, then $\tau\notin\widetilde{L}^0(\sigma)$. By the choice of z, $\mathrm{Supp}(\sigma)\cap\mathcal{C}_l=\emptyset$ for all $z<l\leq r+1$, so if $\tau_0=\sigma$, then $\tau=\sigma^{\rightarrow}_{(p,z)}$ (resp. $\tau=a\sigma^{(p,r+1)}_{(z,h)}$). In both cases, $\tau\in N^0(\sigma)$.

iv) Now, let us consider the case when σ_r, τ_r satisfy (12), $i_l>j_l+1$ for all $1\leq l\leq r$, $p_l>l+1$ for all $l\leq r_0=\min\{r,s(\tau_r)\}$, but $\mathrm{Supp}(\sigma)\cap\mathcal{C}_{r+1}=\{(i,r+1)\}\neq\emptyset$. If $\mathrm{Supp}(\tau)\cap\mathcal{C}_{r+1}=\emptyset$, then σ_{r+1}, τ_{r+1} satisfy (12), so assume $\mathrm{Supp}(\tau)\cap\mathcal{C}_{r+1}=\{(p,r+1)\}\neq\emptyset$. Furthermore, assume $r_0=s(\tau_r)=r$ and $p=p_{r+1}>p_r$ (and so $p<i$). Put

$$\begin{aligned}&\tau_0=(p_1,1)\dots(p_{r-1},r-1)(p,r)(p_r,r+1)(p_{r+2},q_{r+2})\dots(p_t,q_t), \text{ i.e.,}\\&\mathrm{Supp}(\tau_0)=(\mathrm{Supp}(\tau)\setminus\{(p_r,r),(p,r+1)\})\cup\{(p,r),(p_r,r+1)\}.\end{aligned}$$

Obviously, $\tau<^*\tau_0$. If $\tau_0<^*\sigma$, then $\tau\notin\widetilde{L}^0(\sigma)$, because $s(\tau_0)=s(\tau)=t=s(\sigma)$. Since $p_r<p<i$, $\tau_0\neq\sigma$. If $\tau_0\not<^*\sigma$ (i.e., $i_r<p$), then put also

$$\begin{aligned}&\sigma_0=(i_1,j_1)\dots(i_{r-1},r-1)(i,r)(i_r,r+1)(i_{r+2},j_{r+2})\dots(i_t,j_t), \text{ i.e.,}\\&\mathrm{Supp}(\sigma_0)=(\mathrm{Supp}(\sigma)\setminus\{(i_r,r),(i,r+1)\})\cup\{(i,r),(i_r,r+1)\}.\end{aligned}$$

In this case, $\sigma_0\in S_n^2$, $\sigma_0>^*\sigma$, $\sigma_0>^*\tau_0$ and $s(\sigma_0)=t=s(\tau_0)$, so, by the second induction hypothesis, there exists $w_1\in N^0(\sigma_0)\cup N^-(\sigma_0)$ such that $w_1\geq^*\tau_0$. One can check that this implies an existence of $w\in N^0(\sigma)\cup N^-(\sigma)$ such that $w\geq^*\tau$. (In fact, w is obtained from σ by the "same" operation as w_1 from σ_0.)

v) Next, suppose σ_r, τ_r satisfy (12), $i_l>j_l+1$ for all $1\leq l\leq r$, $p_l>l+1$ for all $l\leq r_0=\min\{r,s(\tau_r)\}$, $\mathrm{Supp}(\sigma)\cap\mathcal{C}_{r+1}=\{(i,r+1)\}\neq\emptyset$, $\mathrm{Supp}(\tau)\cap\mathcal{C}_{r+1}=\{(p,r+1)\}\neq\emptyset$, but $r_0=s(\tau_r)=d<r$. If $\mathrm{Supp}(\tau)\cap\mathcal{R}_r=\emptyset$, then put $\tau_0=(p_1,1)\dots(p_d,d)(p,r)(p_{d+2},q_{d+2})\dots(p_t,q_t)$, i.e.,

$$\mathrm{Supp}(\tau_0)=(\mathrm{Supp}(\tau)\setminus\{(p,r+1)\})\cup\{(p,r)\}.$$

At the same time, if $\mathrm{Supp}(\tau)\cap\mathcal{R}_r=\{(r,q)\}\neq\emptyset$, then define τ_0 to be the involution such that

$$\mathrm{Supp}(\tau_0)=(\mathrm{Supp}(\tau)\setminus\{(r,q),(p,r+1)\})\cup\{(r+1,q),(p,r)\}.$$

In both cases, $\tau<^*\tau_0$ and $\tau_0\neq\sigma$. If $\tau_0<^*\sigma$, then $\tau\notin\widetilde{L}^0(\sigma)$, because $s(\tau_0)=s(\tau)=t=s(\sigma)$.

At the contrary, suppose $\tau_0\not<^*\sigma$. If $\mathrm{Supp}(\sigma)\cap(\mathcal{C}_r\cup\mathcal{R}_r)=\emptyset$, then define σ_0 to be the involution such that $\mathrm{Supp}(\sigma_0)=(\mathrm{Supp}(\sigma)\setminus\{(i,r+1)\})\cup\{(i,r)\}$. If $\mathrm{Supp}(\sigma)\cap\mathcal{C}_r=\{(x,r)\}\neq\emptyset$, then $x<r$, so define σ_0 by putting

$$\mathrm{Supp}(\sigma_0)=(\mathrm{Supp}(\sigma)\setminus\{(x,r),(i,r+1)\})\cup\{(i,r),(x,r+1)\}.$$

Finally, if $\mathrm{Supp}(\sigma)\cap\mathcal{R}_r=\{(r,y)\}\neq\emptyset$, then define σ_0 by putting

$$\mathrm{Supp}(\sigma_0)=(\mathrm{Supp}(\sigma)\setminus\{(r,y),(i,r+1)\})\cup\{(r+1,y),(i,r))\}.$$

In all cases, $\sigma_0\in S_n^2$, $\sigma_0>^*\sigma$, $\sigma_0>^*\tau_0$ and $s(\sigma_0)=s(\tau_0)=t$. Hence, by the second inductive assumption, there exists $w_1\in N^0(\sigma_0)\cup N^-(\sigma_0)$ such that $w_1\geq^*\tau_0$. One can check that this implies an existence of $w\in N^0(\sigma)\cup N^-(\sigma)$ such that $w\geq^*\tau$. (In fact, w is obtained from σ by the "same" operation as w_1 from σ_0.) The proof of (13) is complete.

vi) Now, we may assume without loss of generality that σ, τ satisfy (12). First, suppose $i_k = j_k+1$ for some $k \leq t$ and $q_l > l+1$ for all $1 \leq l \leq d < k$. Then $(i_k, j_k) \in M(\sigma)$ and $\tau <^* w = \sigma^-_{(i_k,j_k)} \in N^-(\sigma)$. Second, assume $i_l > j_l + 1$ for all $1 \leq l \leq t$ and $p_l > q_l + 1 = l + 1$ for all l such that $q_l = l \leq j_t$. If $q_t = j_t = t$, then $i_l \geq p_l \geq p_t > t + 1$ for all $1 \leq l \leq t$, so $\mathrm{Supp}(\sigma) \cap \mathcal{R}_{t+1} = \mathrm{Supp}(\tau) \cap \mathcal{R}_{t+1} = \emptyset$. Thus, $\widetilde{\tau} = P_t(\tau)$ and $\widetilde{\sigma} = P_t(\sigma)$ belong to $S_n^2 \cap \widetilde{S}_{n-1}$, so, by the first induction hypothesis, there exists $\widetilde{w} \in N^0(\widetilde{\sigma}) \cup N^-(\widetilde{\sigma})$ such that $\widetilde{\tau} \leq^* \widetilde{w}$. Let w be the unique involution such that $\widetilde{w} = P_t(w)$, then $w \in N^0(\sigma) \cup N^-(\sigma)$ and $\tau \leq^* w$. On the other hand, if $q_t > j_t$, then, arguing as on step iii), one can show that either $\tau \in N^0(\sigma)$ or $\tau \notin \widetilde{L}^0(\sigma)$.

vii) Finally, let us consider the most interesting case when σ, τ satisfy (12), but $p_d = d + 1$ for some $d \leq t$ such that $q_d = d \leq j_t$. Put $\tau_0 = \tau^-_{(d+1,d)}$, then $\tau_0 <^* \tau <^* \sigma$ and $s(\tau_0) = t - 1 < t = s(\sigma)$. By Lemma 3.5, there exists $w_1 \in N^-(\sigma)$ such that $w_1 \geq^* \tau_0$ (in fact, $w_1 >^* \tau_0$). If $w_1 >^* \tau$, then the result follows. Thus, it remains to consider the case $w_1 \not>^* \tau$. This means that $(i_d, d) \in M(\sigma)$ and $w_1 = \sigma^-_{(i_d,d)}$. Denote $\sigma_0 = w_1$ and $\widetilde{\sigma} = P_{d-1}(\sigma_0)$, $\widetilde{\tau} = P_{d-1}(\tau_0)$. We see that $\widetilde{\sigma}, \widetilde{\tau} \in \widetilde{S}_{n-1} \cap S_n^2$, $\widetilde{\sigma} \geq^* \widetilde{\tau}$ and $s(\widetilde{\sigma}) = t - 1 = s(\widetilde{\tau})$. If $\widetilde{\tau} = \widetilde{\sigma}$, then $\mathrm{Supp}(\sigma) \cap \mathcal{C}_{d+1} = \emptyset$ (if $\mathrm{Supp}(\sigma) \cap \mathcal{C}_{d+1} = \mathrm{Supp}(\tau) \cap \mathcal{C}_{d+1} \neq \emptyset$, then (7) doesn't hold for τ, because $(d+1, d) \in \mathrm{Supp}(\tau)$). Moreover, by (12), $i_l \geq p_l > p_d = d+1$ for all $1 \leq l \leq d-1$, so $\mathrm{Supp}(\sigma) \cap \mathcal{R}_{d+1} = \emptyset$. In other words, $\sigma(d+1) = d+1$; in particular,

$$m = \max\{i \mid i_d < i \leq d+1 \text{ and } \sigma(i) = i\}$$

exists. Thus, $w = \sigma^{\uparrow}_{(i_d,d)} \in N^0(\sigma)$ is well-defined and $\tau \leq^* w$.

From now on, assume $\widetilde{\tau} <^* \widetilde{\sigma}$. Then the first inductive assumption shows that there exists $\widetilde{w}_2 \in N^0(\widetilde{\sigma}) \cup N^-(\widetilde{\sigma})$ such that $\widetilde{w}_2 \geq^* \widetilde{\tau}$. Denote $i = i_d$. If $\mathrm{Supp}(\widetilde{w}_2) \cap (\mathcal{C}_i \cup \mathcal{R}_i) = \emptyset$, then denote by w_2 the unique involution such that $\widetilde{w}_2 = P_{d-1}(w_2)$ and define w by putting $\mathrm{Supp}(w) = \mathrm{Supp}(w_2) \cup \{(i, d)\}$. It's easy to see that $w \in N^0(\sigma) \cup N^-(\sigma)$ and $w \geq^* \tau$. On the other hand, suppose $(i, j) \in \mathrm{Supp}(\widetilde{w}_2)$ for some j. If $j < d$, then $\widetilde{w}_2 = \widetilde{\sigma}^{\uparrow}_{(x,j)}$ for some x. (Indeed, if $\widetilde{w}_2 = a\widetilde{\sigma}^{(\alpha,\beta)}_{(x,j)}$ for some $(\alpha, \beta) \in A_{x,j}(\widetilde{\sigma})$, then $\alpha = i$, which contradicts (7).) In this case, put $w = b\sigma^{(\alpha,\beta)}_{(i,d)}$ for some $(\alpha, \beta) \in B_{i,d}(\sigma)$ (since $x > i$, (α, β) exists). It follows from $\widetilde{w}_2 \geq^* \widetilde{\tau}$ that $w \geq^* \tau$.

Next, assume $(i, j) \in \mathrm{Supp}(\widetilde{w}_2)$ for some $j > d$. Denote $(x, j) = \mathrm{Supp}(\sigma) \cap \mathcal{C}_j$ (i.e., $\mathrm{Supp}(\widetilde{w}_2) = (\mathrm{Supp}(\widetilde{\sigma}) \setminus \{(x, j)\}) \cup \{i, j\}$). Since $\widetilde{w}_2$ is well-defined, there are no $(\alpha, \beta) \in \mathrm{Supp}(\sigma)$ such that $(\alpha, \beta) < (x, j)$ and $\alpha > i$. But $(i, d) \in M(\sigma)$, so $\mathrm{Supp}(\sigma) \cap \{(\alpha, \beta) \in \Phi \mid (\alpha, \beta) < (i, d)\} = \emptyset$. Hence $(x, j) \in M(\sigma)$ and $w = \sigma^-_{(x,j)} \geq^* \tau$ (in fact, $w >^* \tau$).

Similarly, assume $(r, i) \in \mathrm{Supp}(\widetilde{w}_2)$ for some $r > i$. Denote $(r, s) = \mathrm{Supp}(\sigma) \cap \mathcal{R}_r$, i.e., $\mathrm{Supp}(\widetilde{w}_2) = (\mathrm{Supp}(\widetilde{\sigma}) \setminus \{(r, s)\}) \cup \{r, i\}$ (it follows from (12) that $s > d$). If $\sigma(a) = a$ for some $i < a < d$, then $w = \sigma^{\uparrow}_{(i,d)}$ is well-defined and $w \geq^* \tau$, so suppose $\sigma(a) \neq a$ for all $i < a < d$. Since $\widetilde{w}_2$ is well-defined and $(i, d) \in M(\sigma)$, there are no $(\alpha, \beta) \in \mathrm{Supp}(\sigma)$ such that $(\alpha, \beta) < (i, d)$ or $(\alpha, \beta) < (r, s)$, $(\alpha, \beta) \not< (r, i)$. Thus, $w = a\sigma^{(r,s)}_{(i,d)}$ is well-defined and $w \geq^* \tau$. The proof is complete. □

Lemma 3.7. *Let $\sigma \in S_n^2$. Then $L^+(\sigma) = N^+(\sigma)$.*

Proof. By Lemma 3.4, it's enough to check that $L^+(\sigma) \subseteq N^+(\sigma)$. By definition,

$$L^+(\sigma) = \{\sigma' \in S_n^2 \mid \sigma' \leq^* \sigma,\ s(\sigma') > s(\sigma),\ \text{and if } \sigma' \leq^* w <^* \sigma,\ \text{then } w = \sigma'\},$$

so it suffice to show that

$$\begin{aligned}&\text{if } \tau \leq^* \sigma \text{ and } s(\tau) > s(\sigma),\\ &\text{then there exists } \sigma' \in N^+(\sigma) \cup N^0(\sigma) \cup N^-(\sigma) \text{ such that } \tau \leq^* \sigma' <^* \sigma.\end{aligned} \tag{14}$$

We will proceed by induction on n (for $n = 1$, there is nothing to prove). The proof is rather long, so we split it into four steps.

i) Note that if $\sigma = w_0$, the maximal element of S_n^2 with respect to $\leq^*$, then there is nothing to prove, because $L^+(w_0) = \emptyset$. Hence we may use the second (downward) induction on $\leq^*$. Let $\sigma = (i_1, j_1) \dots (i_s, j_s) \in S_n^2$, $\sigma <^* w_0$, $\tau = (p_1, q_1) \dots (p_t, q_t) <^* \sigma$ and $s < t$. Consider the following conditions (cf. (9) and (12)):

$$\begin{aligned}
&\text{a) There exists } k \leq s \text{ such that } i_l > l+1 \text{ for all } 1 \leq l \leq k-1.\\
&\text{b) There exists } d \leq k \text{ such that } q_l = l \text{ for all } 1 \leq l \leq d\\
&\quad\ \text{and either } d < k,\ q_{d+1} > k, \text{ or } d = k,\ p_d = d+1.\\
&\text{c) } j_l = l \text{ for all } 1 \leq l \leq d, \text{ and if } d < k, \text{ then either } i_k = j_k + 1 \text{ or } k = s.\\
&\text{d) } (p_l, q_l) = (p_l, l) > (p_{l+1}, q_{l+1}) = (p_{l+1}, l+1), \text{ i.e., } p_l > p_{l+1}, \text{ for all } 1 \leq l \leq d-1.\\
&\text{e) } (i_l, j_l) = (i_l, l) \geq (p_l, q_l) = (p_l, l), \text{ i.e., } i_l \geq p_l, \text{ for all } 1 \leq l \leq d.
\end{aligned} \tag{15}$$

Pick $r \leq s$. Define σ_r and τ_r as in the previous Lemmas. We claim that

$$\begin{aligned}
&\text{if (14) holds for all } \sigma, \tau \text{ satisfying (15),}\\
&\text{then (14) holds for all } \sigma, \tau \in S_n^2.
\end{aligned} \tag{16}$$

Clearly, it's enough to prove that if (14) holds for all σ, τ satisfying (15), and σ_r, τ_r don't satisfy (15) for some $1 \leq r \leq s$, then (14) holds for σ, τ.

We will proceed by induction on r. Evidently, there exist $w_1 = \sigma$, w_2, ..., $w_z = \tau \in S_n^2$ such that $w_1 >^* w_2 >^* \dots >^* w_z$ and $w_{i+1} \in L^+(w_i)$ for all $1 \leq i < z$, so we may assume $\tau \in L^+(\sigma)$. The base $r = 1$ is evident (see step ii) of the proof of Lemma 3.5). The induction step can be performed as on steps iii)–v) of the proof of Lemma 3.6, so assume without loss of generality that σ, τ satisfy (15).

ii) If either $i_k = j_k + 1$ for some $k \leq t$ and $q_l > l+1$ for all $1 \leq l \leq d < k$, or $i_l > j_l + 1$ for all $1 \leq l \leq s$ and $p_l > q_l + 1 = l + 1$ for all l such that $q_l = l \leq j_s$, then, arguing as on step vi) of the proof of Lemma 3.6, we obtain the result. Let us consider the most interesting case when σ, τ satisfy (15), but $p_d = d+1$ for some $d \leq t$ such that $q_d = d \leq j_s$ (and so $j_d = d$, too).

Put $\tau_0 = \tau^-_{(d+1,d)}$, then $\tau_0 <^* \tau <^* \sigma$ and $s(\tau_0) = t-1$. First, suppose $s(\sigma) = s = t-1 = s(\tau_0)$. By Lemma 3.6, there exists $w_1 \in N^0(\sigma) \cup N^-(\sigma)$ such that $w_1 \geq^* \tau_0$. If $w_1 >^* \tau$, then the result follows. Thus, it remains to consider the case $w_1 \not>^* \tau$ (clearly, $w_1 \neq \tau$). Assume $w_1 \in N^0(\sigma)$, then either $w_1 = \sigma^{\to}_{(i_d,d)}$ or $w_1 = a\sigma^{(i_d,d)}_{(x,y)}$ for some $(x, y) \in \mathrm{Supp}(\sigma)$, $y < d$. If $w_1 = \sigma^{\to}_{(i_d,d)}$, then there exists $m > d$ such that $\sigma(m) = m$ and $\sigma(l) \neq l$ for all $d < l < m$. Suppose $l > d+1$, then there exists $j + 1 \leq p < i_d$ such that $(p, m) \in C_{i,j}(\sigma)$, and so $w = c\sigma^{p,m}_{(i,j)} >^* \tau$.

At the contrary, suppose $m = d+1$ and $\sigma(l) \neq l$ for all $d+1 < l < i$. Suppose

$$r = \min\{l \mid d+1 < l < i \text{ and } (z, l) \in \mathrm{Supp}(\sigma) \text{ for some } z < i\}$$

exists (so $(z, r) \in B_{i,j}(\sigma)$). We claim that $w = b\sigma^{(i,j)}_{(z,l)} >^* \tau$. Indeed, it suffice to show that $(R^*_w)_{i,j} \geq (R^*_\tau)_{i,j}$ for all $z < i < i_d$, $d < j < r$. Denote by γ (resp. γ') the number of $(a, b) \in \mathrm{Supp}(\sigma)$ (resp. $(a, b) \in \mathrm{Supp}(\tau)$) such that $b < d$ and $a < i$. Since $\sigma(l) \neq l$ for all $j + 1 < l \leq j$, there are $((j - d - 1) - \gamma)$ (resp. at most $((j - d - 1) - \gamma')$) elements (a, b) in $\mathrm{Supp}(\sigma)$ (resp. in $\mathrm{Supp}(\tau)$) such that $d + 1 < b \leq j$ and $a > i_d$ (resp. $d + 1 < b \leq j$ and $a \geq i$). By (15e), $\gamma \leq \gamma'$. Thus,

$$\begin{aligned}
(R^*_w)_{i,j} - (R^*_\tau)_{i,j} &= \#\{(a, b) \in \mathrm{Supp}(\sigma) \mid b < d \text{ and } a > i\}\\
&\quad - \#\{(a, b) \in \mathrm{Supp}(\tau) \mid b > d+1 \text{ and } a \geq i\}\\
&\geq ((j - d - 1) - \gamma) - ((j - d - 1) - \gamma') = \gamma' - \gamma \geq 0,
\end{aligned}$$

as required. If r doesn't exist, then, arguing as above, one can check that $w = \sigma^{\uparrow}_{(i_d,d)} >^* \tau$. (In fact, $\mathrm{Supp}(w) = (\mathrm{Supp}(\sigma) \setminus \{(i_d, d)\}) \cup \{(d + 1, d)\}$.) The case $w_1 = a\sigma^{(i_d,d)}_{(x,y)}$ is similar.

iii) Next, suppose $w_1 \in N^-(\sigma)$. This means that $(i_d, d) \in M(\sigma)$ and $\mathrm{Supp}(w_1) = \mathrm{Supp}(\sigma)\setminus\{(i_d, d)\}$. Denote $\sigma_0 = w_1$, $\tau_0 = \tau^-_{(d+1,d)}$ and $\widetilde{\sigma} = P_{d-1}(\sigma_0)$, $\widetilde{\tau} = P_{d-1}(\tau_0)$. We see that $\widetilde{\sigma}$, $\widetilde{\tau} \in \widetilde{S}_{n-1} \cap S_n^2$, $\widetilde{\sigma} >^* \widetilde{\tau}$ and $s(\widetilde{\sigma}) = s - 1 < t - 1 = s(\widetilde{\tau})$. The first inductive assumption shows that there exists $\widetilde{w}_2 \in N^+(\widetilde{\sigma}) \cup N^0(\widetilde{\sigma}) \cup N^-(\widetilde{\sigma})$ such that $\widetilde{w}_2 \geq^* \widetilde{\tau}$. If $\widetilde{w}_2 \in N^-(\widetilde{\sigma}) \cup N^0(\widetilde{\sigma})$, then there exists $w \in N^-(\sigma) \cup N^0(\sigma)$ such that $w \geq^* \tau$, see step vi) of the proof of Lemma 3.5 and step vii) of the proof of Lemma 3.6, so it remains to consider the case $\widetilde{w}_2 \in N^+(\widetilde{\sigma})$.

If $\widetilde{w}_2 = c\widetilde{\sigma}^{\alpha,\beta}_{(i,j+1)}$ for some $(i, j) \in \mathrm{Supp}(\sigma)$, $(i, j) > (i_d, d)$, but $(\alpha, j) \not> (i_d, d)$, $(i, \beta) \not> (i_d, d)$, then $d + 1 < \alpha < i_d$ and $\beta \leq d < i_d$, so $\sigma(l) \neq l$ for all $\alpha < l < i_d$. Hence $w = \sigma^{\uparrow}_{(i_d,d)} \geq^* \tau$ (in fact, $\mathrm{Supp}(w) = (\mathrm{Supp}(\sigma)\setminus\{(i_d, d)\})\cup\{(\alpha, d)\}$). On the other hand, if $\widetilde{w}_2 = c\widetilde{\sigma}^{i_d,\beta}_{(i,j)}$ for some $(i, j) \in \mathrm{Supp}(\sigma)$, then $i > i_d$ and $j > d$, so $w = \sigma^{\rightarrow}_{(i,j)} \geq^* \tau$ (in fact, $\mathrm{Supp}(w) = (\mathrm{Supp}(\sigma) \setminus \{(i, j)\}) \cup \{(i, \beta)\}$). If $\widetilde{w}_2 = c\widetilde{\sigma}^{\alpha,i_d}_{(i,j)}$ for some $(i, j) \in \mathrm{Supp}(\sigma)$, then $i > i_d$, $j > d$ and one can easily check that $(i, j) \in A_{(i_d,d)}(\sigma)$, so $w = a\sigma^{(i,j)}_{(i_d,d)} \geq^* \tau$. In all other cases, $w \geq^* \tau$, where $\mathrm{Supp}(w) = \mathrm{Supp}(w_2) \cup \{(i_d, d)\}$ and w_2 is the unique involution such that $\widetilde{w}_2 = P_{d-1}(w_2)$.

iv) Finally, assume $s(\sigma) = s < t - 1 = s(\tau_0)$ (see the beginning of step ii)). Suppose

$$r = \min\{l \mid (z, l) \in \mathrm{Supp}(\tau),\ l > d + 1 \text{ and } z \leq i_d\}$$

exists. Then $\sigma \geq^* \tau_1 >^* \tau$, where

$$\mathrm{Supp}(\tau_1) = (\mathrm{Supp}(\tau) \setminus \{(d, d + 1), (z, r)\}) \cup \{(z, d)\},$$

so $\tau \notin L^+(\sigma)$. At the same time, if r doesn't exists, then define σ_0 to be the involution such that $\mathrm{Supp}(\sigma_0) = \mathrm{Supp}(\sigma) \setminus \{(i_d, d)\}$. In this case, $\sigma_0 \geq^* \tau_0$ and $s(\sigma_0) = s - 1 < t - 1 = s(\tau_0)$, so we may proceed as on the previous step. The proof is complete. □

Finally, we will prove the fact used in the proof of Theorem 1.10.

Lemma 3.8. *Let $\sigma \in S_n^2$. Then $L'(\sigma) = N'(\sigma)$.*

Proof. Pick an involution $\tau = \sigma(i, j)^- \in N^-(\sigma) = L^-(\sigma)$. If $\tau \notin N'(\sigma)$, then there exists m such that $j \leq m \leq i$ and $\sigma(m) = m$. Assume, for example, that $i \neq m$, then $\tau <^* w <^* \sigma$, where

$$\mathrm{Supp}(w) = \mathrm{Supp}(\tau) \cup \{(i, m)\},$$

so $\tau \notin L'(\sigma)$. The case $j \neq m$ is similar.

On the other hand, suppose $\tau \notin L'(\sigma)$. Then there exists $w \in L^+(\sigma)\cup L^0(\sigma)$ such that $\tau <^* w <^* \sigma$. By Lemmas 3.7 and 3.6, $L^+(\sigma) = N^+(\sigma)$ and $L^0(\sigma) = N^0(\sigma)$ respectively. By (4),

$$\mathrm{Supp}(\tau) \cap Y = \mathrm{Supp}(w) \cap Y = \mathrm{Supp}(\sigma) \cap Y,$$

where $Y = \{(p, q) \in \Phi \mid (p, q) \not\leq (i, j)\}$, so $\mathrm{Supp}(w) \cap \widetilde{Y} \neq \emptyset$, where $\widetilde{Y} = \{(p, q) \in \Phi \mid (p, q) \leq (i, j)\}$. It follows from this fact and the definitions of $N^+(\sigma)$ and $N^0(\sigma)$ (see Subsection 2.1) that there exists m such that $j \leq m \leq i$ and $\sigma(m) = m$, so $\tau \notin N'(\sigma)$. (For example, if $w = \sigma^{\uparrow}_{(\alpha,\beta)}$ for some $(\alpha, \beta) \in \mathrm{Supp}(\sigma)$, then $\mathrm{Supp}(w) \setminus \mathrm{Supp}(\sigma) = \{(m, \beta)\}$.) This concludes the proof. □

4. Concluding remarks

4.1. Let $\sigma \in S_n^2$. Using results of [Pa], one can easily obtain a formula for the dimension of the orbit Ω_σ. Let $\xi\colon \mathrm{Supp}(\sigma) \to \mathbb{C}^\times$ be a map. Recall the notation from Subsection 1.2. As above, let $l(\sigma)$ be the length of a reduced expression of σ as a product of simple reflections, and $s(\sigma) = |\mathrm{Supp}(\sigma)|$ (obviously, if $\mathrm{Supp}(\sigma) = \{(i_1, j_1), \ldots, (i_t, j_t)\}$, then $s(\sigma) = t$). By [Pa, Theorem 1.2], $\Theta_{\sigma,\xi}$ is an

irreducible affine variety of dimension $\dim \Theta_{\sigma,\xi} = l(\sigma) - s(\sigma)$. By Lemma 2.1, $\Omega_\sigma = \bigcup_\xi \Theta_{\sigma,\xi}$. Denote $\Theta_0 = \Theta_{\sigma,\xi_0}$, where $\xi_0(\alpha) = 1$ for all $\alpha \in \mathrm{Supp}(\sigma)$ (in other words, Θ_0 is the U-orbit of X_σ^t).

Proposition 4.1. *Let $\sigma \in S_n$ be an involution. Then $\dim \Omega_\sigma = l(\sigma)$.*

PROOF. Let $Z = \mathrm{Stab}_B X_\sigma^t$ be the stabilizer of X_σ^t in B. One has

$$\dim \Omega_\sigma = \dim B - \dim Z.$$

Recall that $B = U \rtimes D$. Suppose $g = ud \in Z$, where $u \in U$, $d \in D$, then $g.X_\sigma^t = u.(d.X_\sigma^t) = X_\sigma^t$. But $d.X_\sigma^t = f_{\sigma,\xi}$, where $\xi_l = d_{i_l,i_l}/d_{j_l,j_l}$ (cf. the proof of Lemma 2.1), so $g.X_\sigma^t = u.f_{\sigma,\xi} \in \Theta_{\sigma,\xi}$. Since $\Theta_{\sigma,\xi} \neq \Theta_0$ for $\xi \neq \xi_0$, we conclude that $\xi = \xi_0$, so $d.X_\sigma^t = X_\sigma^t$. Hence $d \in Z_D$ and $u \in Z_U$, where $Z_D = \mathrm{Stab}_D X_\sigma^t$ (resp. $Z_U = \mathrm{Stab}_U X_\sigma^t$) is the stabilizer of X_σ^t in D (resp. in U).

Since $B = U \rtimes D$ as algebraic groups, the maps

$$\phi\colon B \to U \times D\colon g = ud \mapsto (u,d) \text{ and}$$
$$\psi\colon U \times D \to B\colon (u,d) \mapsto ud$$

are inverse isomorphisms of algebraic varieties. We checked that $\phi(Z) \subseteq Z_U \times Z_D$. The inclusion $\psi(Z_U \times Z_D) \subseteq Z$ is evident, so $\phi(Z) = Z_U \times Z_D$. Thus, $\dim Z = \dim Z_U + \dim Z_D$.

But $d \in D$ belongs to Z_D if and only if $\xi = \xi_0$, i.e., $d_{i_l,i_l} = d_{j_l,j_l}$ for all $1 \le l \le t$. Hence

$$\dim Z_D = \dim D - |\mathrm{Supp}(\sigma)| = n - s(\sigma).$$

On the other hand, since $\dim \Theta_0 = l(\sigma) - s(\sigma)$, we obtain

$$\dim Z_U = \dim U - \dim \Theta_0 = \dim B - n - l(\sigma) + s(\sigma).$$

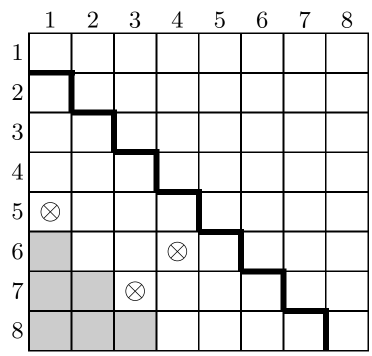
We conclude that $\dim Z = \dim B - l(\sigma)$, and so $\dim \Omega_\sigma = l(\sigma)$, as required. □

4.2. Here we present a conjectural description of the closure of a given B-orbit Ω_σ, $\sigma \in S_n^2$. Namely, we describe a subvariety $Z_\sigma \subseteq \mathfrak{n}^*$, prove that $\overline{\Omega}_\sigma \subseteq Z_\sigma$ and prove the equality $\overline{\Omega}_\sigma = Z_\sigma$ in some particular cases. Define S_σ to be the set of maximal elements from $\mathrm{Supp}(\sigma)$ with respect to the order $\le$ on Φ. Let

$$\mathcal{M}_\sigma = \{\alpha \in \Phi \mid \alpha > \beta \text{ for some } \beta \in S_\sigma\}.$$

Example 4.2. i) Let $\sigma = w_0 = (n,1)(n-1,2)\ldots(n-n_0+1,n_0)$, $n_0 = [n/2]$, be the maximal element of S_n^2 with respect to $\le_B$. Then $S_\sigma = \{(n,1)\}$ and $\mathcal{M}_\sigma = \emptyset$.

ii) Let $n = 8$, $\sigma = (5,1)(7,3)(6,4)$. Then $S_\sigma = \{(5,1),(7,3)\}$ and $\mathcal{M}_\sigma = \{(6,1), (7,1), (8,1), (7,2), (8,2), (8,3)\}$ (these elements are grey on the picture below).



Let $A \in \mathfrak{n}^*$. To each $(r,s) \in \Phi$ one can assign polynomial $\gamma_{r,s}$ in $A_{i,j}$ of the form

$$\gamma_{r,s}(A) = (A^2)_{r,s} = \sum_{k=1}^{n} A_{r,k}A_{k,s} = \sum_{k=s+1}^{r-1} A_{r,k}A_{k,s}.$$

Denote by Z_σ the subvariety of $\mathfrak{n}^*$ defined by

$$\begin{aligned} &\operatorname{rk}\pi_{i,j}(A) \leq (R_\sigma^*)_{i,j} \text{ for all } (i,j) \in \Phi,\\ &\gamma_{i,j}(A) = 0 \text{ for all } (i,j) \in \mathcal{M}_\sigma. \end{aligned} \tag{17}$$

Proposition 4.3. *Let $\sigma \in S_n$ be an involution. Then $\overline{\Omega}_\sigma \subseteq Z_\sigma$.*

PROOF. Let $A \in \overline{\Omega}_\sigma$. Lemma 2.2 guarantees that A satisfies $\operatorname{rk}\pi_{i,j}(A) \leq (R_\sigma^*)_{i,j}$ for all $(i,j) \in \Phi$, so it remains to check that $\gamma_{i,j}(A) = 0$ for all $(i,j) \in \mathcal{M}_\sigma$. Pick an element $(r,s) \in \mathcal{M}_\sigma$. Suppose $A \in \Omega_\sigma$. Recall that there exists $g \in B$ such that $A = (gX_\sigma^t g^{-1})_{\text{low}}$, the strictly lower-triangular part of $y = gX_\sigma^t g^{-1}$ (see Subsection 1.3). But $(X_\sigma^*)^2 = 0$, so $y^2 = 0$. In particular,

$$(y^2)_{r,s} = \sum_{k=1}^{n} y_{r,k}y_{k,s} = \sum_{k=1}^{s} y_{r,k}y_{k,s} + \gamma_{r,s}(A) + \sum_{k=r}^{n} y_{r,k}y_{k,s} = 0.$$

Clearly, $(R_\sigma^*)_{i,j} = 0$ for all $(i,j) \in \mathcal{M}_\sigma$. Hence $y_{r,k} = A_{r,k} = 0$ for all $1 \leq k \leq s$, because if $1 \leq k \leq s$, then $(r,k) > (r,s) \in \mathcal{M}_\sigma$, and so $(r,k) \in \mathcal{M}_\sigma$. This implies $\sum_{k=1}^{s} y_{r,k}y_{k,s} = 0$. Similarly, $y_{k,s} = A_{k,s} = 0$ for all $r \leq k \leq n$, because if $r \leq k \leq n$, then $(k,s) > (r,s) \in \mathcal{M}_\sigma$, and so $(k,s) \in \mathcal{M}_\sigma$. This implies $\sum_{k=r}^{n} y_{r,k}y_{k,s} = 0$. Thus, $\gamma_{r,s}(A) = 0$ for all $A \in \Omega_\sigma$, and so for all $A \in \overline{\Omega}_\sigma$. □

Conjecture 4.4. *Let $\sigma \in S_n$ be an involution. Then $\overline{\Omega}_\sigma = Z_\sigma$.*

Remark 4.5. Suppose $\tau \leq^* \sigma$. Then $\mathcal{M}_\sigma \subseteq \mathcal{M}_\tau$, so that one can see immediately $\gamma_{r,s}(A) = 0$ for $(r,s) \in \mathcal{M}_\sigma$ and $A \in \Omega_\tau$.

Unfortunately, we can neither prove the irreducibility of Z_σ nor compute its dimension. On the other hand, in some particular cases the proof of the equality $\overline{\Omega}_\sigma = Z_\sigma$ is more or less straightforward. Namely, assume $\operatorname{Supp}(\sigma) = \{(i_1,j_1),\ldots,(i_t,j_t)\}$ is a *chain*, i.e., $(i_1,j_1) > \ldots > (i_t,j_t)$. (For instance, $\sigma = w_0$, or, more generally, $\sigma = (n,1)(n-1,2)\ldots(n-k+1,k)$ is maximal among all involutions with $k \leq n_0 = [n/2]$ disjoint cycles.)

Proposition 4.6. *If $\operatorname{Supp}(\sigma)$ is a chain, then $\overline{\Omega}_\sigma = Z_\sigma$.*

PROOF. In this case, $S_\sigma = \{(i_1,j_1)\}$, so

$$\mathcal{M}_\sigma = \{(i,j) \in \Phi \mid i \geq i_1 \text{ and } j \leq j_1\} \setminus \{(i_1,j_1)\}.$$

Suppose $A \in Z_\sigma$. Obviously, if $i > i_1$ or $j < j_1$, then $(R_\sigma^*)_{i,j} = 0$, hence $\operatorname{rk}\pi_{i,j}(A) = 0$ and so $A_{i,j} = 0$. It follows that if $(r,s) \in \mathcal{M}_\sigma$, then $\gamma_{r,s}(A) = 0$. Thus, $A \in \mathfrak{n}^*$ belongs to Z_σ if and only if $\operatorname{rk}\pi_{i,j}(A) \leq (R_\sigma^*)_{i,j}$ for all $(i,j) \in \Phi$.

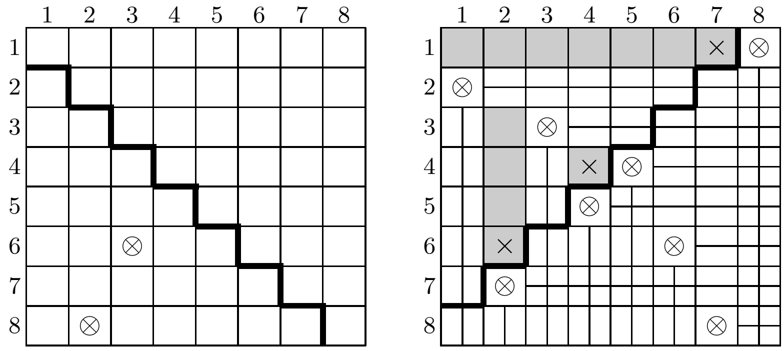
We need some more notation. For $1 \leq i,j \leq n$, let $\widehat{\pi}_{i,j}\colon \mathfrak{g} \to \mathfrak{g}$ be the map sending a matrix $y \in \mathfrak{g} = \operatorname{Mat}_n(\mathbb{C})$ to its upper-left $i \times j$ submatrix. Denote also by $P\colon \mathfrak{g} \to \mathfrak{g}$ the map defined by $P(y)_{i,j} = y_{n-j+1,i}$, $y \in \mathfrak{g}$, $1 \leq i,j \leq n$. (Note that $\widehat{\pi}_{i,j} = \pi_{i,j} \circ P$ for $(i,j) \in \Phi$.) Put $w = w_0\sigma$ and

$$\begin{aligned} &\mathcal{D}(w) = \{(i,j) \mid w(i) > j \text{ and } w^{-1}(j) > i\},\\ &\mathcal{E}(w) = \{(i,j) \in \mathcal{D}(w) \mid (i+1,j) \notin \mathcal{D}(w) \text{ and } (i,j+1) \notin \mathcal{D}(w)\},\\ &Z' = \{y \in \mathfrak{g} \mid \operatorname{rk}\widehat{\pi}_{i,j}(y) \leq \operatorname{rk}\widehat{\pi}_{i,j}(\dot{w}) \text{ for all } 1 \leq i,j \leq n\},\\ &Z = Z' \cap P(\mathfrak{n}^*) = \{y \in Z' \mid y_{i,j} = 0 \text{ for all } i \geq n-j+1\},\\ &Z'' = \{y \in \mathfrak{g} \mid \operatorname{rk}\widehat{\pi}_{i,j}(y) \leq \operatorname{rk}\widehat{\pi}_{i,j}(\dot{w}) \text{ for all } (i,j) \in \mathcal{E}(w)\}. \end{aligned}$$

(Clearly, $Z_\sigma = P(Z)$, because $\operatorname{rk}\widehat{\pi}_{i,j}(\dot{w}) = \operatorname{rk}\pi_{i,j}(P(\dot{w})) = (R_\sigma^*)_{n-j+1,i}$.) For example, if $n = 8$, $\sigma = (8,2)(6,3)$, then $w = (8,7,2,1)(5,4)$ (here we write w as a product of disjoint cycles),

$$\begin{aligned} \mathcal{D}(w) = \{&(1,2),(1,3),(1,4),(1,5),(1,6),\\ &(1,7),(3,2),(4,2),(4,4),(5,2),(6,2)\},\\ \mathcal{E}(w) = \{&(1,7),(4,4),(6,2)\}. \end{aligned}$$

On the left (resp. right) picture below we draw X_σ^* (resp. $\dot{w}$) as a rook placement. On the right picture boxes from $\mathcal{D}(w)$ are grey and boxes from $\mathcal{E}(w)$ are marked by $\times$'s.



Evidently, $Z' \subseteq Z''$. (In [MS], Z' is called a *determinantal matrix Schubert variety.*) But it follows from [MS, Theorem 15.15] that $Z' = Z''$. Furthermore, [MS, Theorem 15.31] claims that Z' is a smooth irreducible affine subvariety of $\mathfrak{g}$ of dimension $\dim Z' = n^2 - l(w)$. Denote

$$V = \{y \in \mathfrak{g} \mid y_{i,j} = 0 \text{ for all } i < n - j + 1\}.$$

Since $\mathrm{Supp}(\sigma)$ is a chain, $\mathcal{D}(w)$ is contained in the set $\{(i,j) \mid i < n - j + 1\}$. This means that $Z' \cong Z \times V$ as affine varieties, hence Z is a smooth irreducible affine variety of dimension

$$\dim Z = \dim Z' - \dim V = \dim \mathfrak{n}^* - l(w).$$

Thus, $Z_\sigma = P(Z)$ is an irreducible subvariety of $\mathfrak{n}^*$ of dimension $\dim Z = \dim \mathfrak{n}^* - l(w)$. But it is well-known that $l(w) = l(w_0\sigma) = |\Phi| - l(\sigma) = \dim \mathfrak{n}^* - l(\sigma)$, so $\dim Z_\sigma = l(\sigma)$. Finally, $\overline{\Omega}_\sigma \subseteq Z_\sigma$ (by Proposition 4.3) and $\dim \overline{\Omega}_\sigma = l(\sigma)$ (by Proposition 4.1), so $\overline{\Omega}_\sigma = Z_\sigma$. $\square$

Note that Conjecture 4.4 together with Conjecture 1.11 imply an explicit description of the tangent cone C_σ to the Schubert variety X_σ for an involution σ.

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